When graph meets diagonal: an approximative access to fixed point theory

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Introduction

Two of the most fundamental fixed point theorems are those of S. Banach [Ban22]: *every contractive function* $f : X \longrightarrow X$ *on a complete metric space* X *has a fixed point,* and L. E. J. Brouwer [Bro12]: *every continuous function* $f : P \longrightarrow P$ *on a compact convex subset* P *of a finite dimensional Hausdorff topological vector space* E *(short hand for* convex polyhedron*) has a fixed point.*

The second theorem gives rise to one of the greatest challenging problems in fixed point theory which is known as *Schauder's Conjecture* [Sch30], [Me81, Problem 54].

Does every continuous function $f : K \longrightarrow K$ *have a fixed point provided* K *is a compact convex subset of a HTVS* E*?*

In 1935, using Brouwer's fixed point theorem, A. Tychonoff [Tyc35] gave an affirmative answer for the class of locally convex spaces E. He made use of a uniform approximation technique which can be understood as a factorization/selection result

$$\begin{tikzcd} K \arrow[rr, "Vf"] \arrow[dr, "id_V f"'] & & K \\ & P \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{0.1}$$

where P is a convex polyhedron, V a vicinity in $K \times K$ which is identified with the graph of a (convex-valued) map $V : K \longrightarrow 2^K$ and id_V is a continuous displacement of the identity which graph is contained in V. There exists a fixed point x_V of $(id_V f \iota)$ by Brouwer's theorem, hence $\iota(x_V)$ is a fixed point of (Vf) by the factorization (0.1). Since the graph of f is closed the existence of a fixed point of f follows now straightforwardly by means of a convergent subnet of $(\iota(x_V))_V$ which exists in view of compactness of K.

In 1941 S. Kakutani [Kak41] generalized Brouwer's result to convex-valued maps $F : P \longrightarrow 2^P$ which are closed, i. e. their graphs are closed. His result is based on a selection technique and holds for convex compacta K in Fréchet spaces, too.

The two major arguments which lead to Tychonoff's result are therefore on the one hand the existence of a finite dimensional approximation id_V, which is closely related to the local convexity of E. On the other hand the existence of a subnet of $(\iota(x_V))_V$, which converges to a fixed point of f, is essential. Here compactness of K and closedness of the graph of f come into operation. Generalizations of this arguments were the starting-points of various new fixed point results. See e. g. [Had84] for an outline until 1984.

In 1977 J. W. Roberts [Rob77b], [Rob77a] gave examples of compact convex subsets K_R of non-locally convex HTVSs E where the Krein-Milman theorem fails. His construction was the first promising candidate for a counterexample to Schauder's conjecture. In 1984 N. J. Kalton, N. T. Peck and J. W. Roberts [KPR84] stated (without proof) that these spaces K_R enjoy a quality they christened *simplicial approximation property*. It basically means that merely the graph of the restricted approximation $(\iota\, id_V f)|_{(\iota P)}$ in (0.1) is contained in the graph of $(V f)$. In 1994 N. T. Nhu and L. H. Tri [NT94] proved the simplicial approximation property for the spaces K_R from which an affirmative answer to Schauder's Conjecture for this class of spaces immediately follows.

Prompted by the spaces K_R we make the following observations regarding the factorization/selection (0.1).

1st. There is no need for P to be a subset of K, i. e. ι need not be an embedding. 2nd. The concatenation $(\iota\, id_V f)$ must approximate f only on their fixed points. 3rd. The existence of a convergent subnet of $(\iota(x_V))_V$ need not come from a compactness argument.

So far observe that the Banach theorem can be reinterpreted by such a weakened factorization, too. In fact, if x_0 is the (unique) fixed point of $f : X \longrightarrow X$ then

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "x_0"'] & & X \\ & \{x_0\} \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

where x_0 denotes the constant function $x \mapsto x_0$. I. e. $(\iota\, x_0)$ approximates f only on its fixed point(s).

Two of the main problems in fixed point theory are existence and location of fixed points. One of the most elementary concepts in order to treat this problems is *Topological Transversality* which goes back to 1936 and K. Borsuk [Bor36] and was extensively elaborated by A. Granas [Gra59], [Gra62] starting from 1959. An axiomatic approach to this theory was given by T. Jerofsky [Jer82] in 1982.

We give here a short outline of Topological Transversality. Let (B, A) be a pair in a Tychonoff space X, i. e. A and B are closed subsets of X and $A \subseteq B$. Consider a fixed class $\mathcal{X}$ of closed (set-valued) maps $F : B \longrightarrow 2^X$ which are fixed point-free on A. We partition $\mathcal{X}$ into two subclasses: those $F \in \mathcal{X}$ where no fixed point-free $\tilde{F} \in \mathcal{X}$ exists which coincides with F on A and the complement of this class. Usually these subclasses are called the *essential* and the *inessential* maps, respectively. Consider now a closed homotopy $F^t : B \longrightarrow 2^X$, $t \in I$ $(I = [0, 1])$, i. e. the map $(t, x) \mapsto F^t(x)$ is closed. Suppose that F^0 is fixed point-free and the set of fixed points $\{x \in B\,;\, x \in F^t(x) \text{ for some } t \in I\}$ of the homotopy is a compact subset of B which is disjoint to A. Then there is an Urysohn function $\mu : B \longrightarrow I$ which is equal to 1 on A and vanishes on the fixed points of F^t. Hence $F^\mu : B \longrightarrow 2^X$ is a closed fixed point-free map which coincides with F^1 on A and, provided F^μ belongs to $\mathcal{X}$, F^1 turns out to be inessential.

For historical reasons this technique is called *Homotopy Extension*. In general it applies indirectly whenever some essential maps are known: suppose F^1 is essential and F^μ belongs to $\mathcal{X}$; then we infer the alternative that either F^0 has a fixed point or the graphs of the F^t must *traverse* the diagonal in $A \times A$ for some $t \in I$. The existence of essential maps is closely related fixed point results, see e. g. [DG82, Chapter II].

Of fundamental importance for Topological Transversality is the determination of conditions under which a map $F^\mu : B \longrightarrow 2^X$ belongs to the class $\mathcal{X}$ provided $F^t : B \longrightarrow 2^X$ is a closed homotopy and $\mu : B \longrightarrow I$ a continuous functional. Even though there is no Homotopy Extension Theorem available the concept of Topological Transversality holds for the class $\mathcal{X}$ of contractive functions $f : X \longrightarrow X$ on complete metric spaces (X, d), too. A class of homotopies which work well with contractive functions are *α-contractive families* $f^t : X \longrightarrow X$, i. e.

$$d\left(f^t(x_1), f^t(x_2)\right) \leq \alpha d(x_1, x_2), \quad x_1, x_2 \in X, \quad t \in I$$

and

$$d\left(f^{t_1}(x), f^{t_2}(x)\right) \leq M|t_1 - t_2|, \quad x \in X, \quad t_1, t_2 \in I$$

hold for some $0 \leq \alpha < 1$ and $M \geq 0$. The set of fixed points of an α-contractive family forms an arc $t \mapsto x_t$ in X. To investigate Topological Transversality is closely related to determine the component of $X \setminus A$ to which the above arc belongs. See e. g. [Gra94].

Recalling our observations regarding the triangle (0.1) one observes that we are again in position to draw a weakened triangle

$$\begin{tikzcd}
X \arrow[rr, "f^t"] \arrow[dr, "t"'] & & X \\
& I \arrow[ur, hook, "x_t"'] &
\end{tikzcd}$$

In what follows we present a general concept of ‘triangles’ like the above ones We consider uniform spaces X and closed (set-valued) maps $F : X \longrightarrow 2^X$ and introduce a class $\mathcal{X}$ of regular maps which enjoy the fixed point property and include both: convex-valued maps on convex compacta in Fréchet spaces and (bounded) contractive functions on complete metric spaces. This class $\mathcal{X}$ generalizes naturally to a class of regular homotopies appropriate to derive a Homotopy Extension Theorem. We develop Topological Transversality in the context of contractible and locally contractible metrizable spaces X. A Nonlinear Alternative and a Sweeping Theorem are stated for illustration.

SECTION 1 starts with some notation regarding uniform spaces, Hausdorff topological vector spaces and polyhedra where in particular convex polyhedra are introduced. What follows are classical selection, factorization, approximation and extension results which are slightly customized to our purpose. We close with a list of some well-known fixed point theorems for later reference.

SECTION 2 provides our major concepts of *proximity* and *preciseness* which meet in the notion of *approximability* that will play the role of (0.1) in what follows. The subsequently defined *subnet condition* is the corresponding substitute for compactness. Approximability and the subnet condition themself meet in the notion of *regular (set-valued) maps* $F : X \longrightarrow 2^X$. So far regularity turns out to be rather a quality of graphs than of spaces and maps. Regular maps always meet the diagonal, i. e. they have a fixed point. We generalize the definition of regularity to maps $F : A \longrightarrow 2^X$ defined on closed subsets A of X. Various accompanying examples are given which in particular contain the classical ones: (bounded) contractive functions on complete metric spaces, compact upper semicontinuous convex-valued maps on Fréchet spaces, continuous functions on compact absolute retracts. In particular constant functions turn out to be regular.

SECTION 3 introduces *regular homotopies* $F^t : A \longrightarrow 2^X$ which are - at least for compact spaces X - understood as closed deformations of regular maps. Arcs turn out to be the generalization of the constants. Further examples are bounded *α-contractive families*, compact upper semicontinuous convex-valued homotopies on Fréchet spaces and continuous homotopies on compact absolute retracts. After a general investigation of regular homotopies we examine conditions on that $F^\mu : A \longrightarrow 2^X$ is a regular map provided $F^t : A \longrightarrow 2^X$ is a regular homotopy and $\mu : A \longrightarrow I$ a continuous functional. We suppose that μ is constant in a neighborhood of the fixed points of F^t which is, in view of the Homotopy Extension technique, natural. Next we consider closed pairs (B, A) in X and introduce *A-regular homotopies* $F^t : B \longrightarrow 2^X$ with respect to a finite subset I' of I. It basically states that we are in position to approximate the homotopy $F^t : B \longrightarrow 2^A$ uniformly on A whenever t belongs to I'. Now we are in position to consider *essentiality* and *inesssentiality* and to give necessary and sufficient conditions for constant functions to be essential. We prove an appropriate *Homotopy Extension Theorem* from which a *Nonlinear Alternative* and a *Sweeping Theorem* in contractible, locally contractible and metrizable spaces and topological groups, respectively, are derived.

SECTION 4 comes back to our motivating Roberts spaces K_R. We recapitulate the construction of convex compacta having no extreme points and define in particular the notion of *needle points* in HTVS. The K_R's turn out to be spaces with *simplicial approximation property* or, more restrictive, *weakly admissible* spaces, which are introduced in the sequel. The Kakutani fixed point theorem and the Nonlinear Alternative for spaces with simplicial approximation property complete this section.

SECTION 5 examinates further fixed point theorems and their relationship to regularity. *Non-expansive functions* fail to be regular but more subtle modification of contractiveness leads again to regularity: we show regularity of *weakly contractive families* which go back to J. Dugundji and A. Granas [DG78].

Due to C. D. Horvath [Hor91] are Φ*-spaces* which are understood to be generalizations of locally convex spaces. Compact continuous homotopies in Φ-spaces turn out to be regular.

At the end of this section we point out some interconnections between our previous examples, Φ-spaces, admissible spaces and absolute retracts and obtain a characterization of absolute retracts in σ-compact convex subsets of metrizable HTVS which generalizes the characterization given by T. Dobrowolski [Dob85].

SECTION 6 gives a short list of problems which arose during our investigations and are - to the authors mind - of interest.

1 Preliminaries

This chapter provides basic definitions and statements needed almost over all what follows. Some familiarity with general topology and linear spaces is assumed, see e. g. [Dug73], [Kel55] and [Köt66], respectively. Homological terms are only needed for Theorem 1.13 and [ES52], [Spa66] are standard references. More special terms, closely related to particular sections, are given when needed.

For some of the following concepts there exists a 'variety of standard' notation in the literature. The following one is customized for our purpose.

The *one-point space* is denoted by $\{*\}$, the *unit interval* $[0, 1]$ by I, the n-dimensional *cubes*, i. e. n-times products of I, by I^n, the *Hilbert cube* by I^∞, the n-dimensional *cells*, i. e. the interiors of the above cubes, by D^n , the n-dimensional *spheres*, i. e. the boundaries of the cubes I^{n+1}, by S^n.

For any subset A of a topological space we denote by $\mathring{A}$, $\overline{A}$ and ∂A *interior*, *closure* and *boundary* of A, respectively.

If we use these terms relative to a subspace B of the underlying topological space the above symbols are indexed by B. E. g. $\partial_B A$ denotes the boundary of some subset A of B relative B.

The *join* of two topological spaces X and Y is the quotient space of $X \times I \times Y$ by the relation $(x, t, y) \sim (x', t', y')$ if $(t = t' = 0$ and $x = x'$ or $t = t' = 1$ and $y = y'$ or $(x, t, y) = (x', t', y'))$ and denoted by $X \star Y$. The *cone* over a space X is given by $X \star \{*\}$ and denoted by $\mathit{cone}\, X$.

Furthermore *id* and ι always denote the *identity* and *embeddings*, respectively, anyway what spaces are considered. Finally by $\mathbb{N}$ and $\mathbb{R}$ we denote the positive integers and the real numbers, respectively.

1.1 Uniform spaces

Let X be a set. We denote by $\Delta(X)$ the diagonal $\{(x, x)\,;\, x \in X\}$ in $X \times X$. For subsets U and V of $X \times X$ let $VU := \{(x, z) \in X \times X\,;\, (x, y) \in U, (y, z) \in V$ for some $y \in X\}$ and $V^{-1} := \{(x, y)\,;\, (y, x) \in V\}$.

A non-empty system $\mathcal{V}(X)$ of subsets of $X \times X$ is said to be a *uniformity* for X if

(i) each element of $\mathcal{V}(X)$ contains the diagonal $\Delta(X)$,

(ii) $V \in \mathcal{V}(X)$ iff $V^{-1} \in \mathcal{V}(X)$,

(iii) for each $V \in \mathcal{V}(X)$ there exists $U \in \mathcal{V}(X)$ such that $UU \subseteq V$,

(iv) $U \in \mathcal{V}(X)$, $V \in \mathcal{V}(X)$ implies $U \cap V \in \mathcal{V}(X)$ and

(v) $U \in \mathcal{V}(X)$, $U \subseteq V$ implies $V \in \mathcal{V}(X)$.

The pair $(X, \mathcal{V}(X))$ is called a *uniform space*. The elements of $\mathcal{V}(X)$ are called *vicinities*. If $V = V^{-1}$ then V is said to be *symmetric*. A subfamily $\mathcal{V}'(X)$ of $\mathcal{V}(X)$ is said to be a *base* for the uniformity $\mathcal{V}(X)$ if each member of $\mathcal{V}(X)$ contains a member of $\mathcal{V}'(X)$. A subfamily $\mathcal{V}'(X)$ of $\mathcal{V}(X)$ is said to be a *subbase* for the uniformity $\mathcal{V}(X)$ if the family of all finite intersections of members of $\mathcal{V}'(X)$ is a base for the uniformity $\mathcal{V}(X)$. A uniformity is said to be *Hausdorff* if $\Delta(X) = \cap_{V \in \mathcal{V}(X)} V$. Each Hausdorff uniformity has a base of symmetric vicinities whose intersection is the diagonal.

A uniformity $\mathcal{V}(X)$ assigns to each $x \in X$ a system of neighborhoods $(V(x))_{V \in \mathcal{V}(X)}$ by $V(x) := \{y \in X\,;\,(x,y) \in V\}$ which determines a topology τ for X. The induced topological space (X, τ) is Hausdorff iff the uniformity is Hausdorff. If the diagonal $\Delta(X)$ belongs to the uniformity $\mathcal{V}(X)$ the uniform space is said to be *discrete* and the induced topology is discrete. A uniform space X is *metrizable* if it is Hausdorff and has a countable base for its uniformity $\mathcal{V}(X)$. For a metrizable uniform space there exists a metric d on X which induces the same topology on X as the uniformity $\mathcal{V}(X)$ and we call X a *metric space* (X, d) If the metric itself is of minor interest their induced ε-vicinities are written as $V_\varepsilon := \{(x,y) \in X \times X\,;\,(x,y) < \varepsilon\}$.

Consider a Hausdorff topological space X and two disjoint closed subsets A and B of X. A function $\mu : X \longrightarrow I$ is said to be an *Urysohn function* with respect to A and B if μ is continuous, equals 1 on A and vanishes on B. X is said to be *completely regular* (or *Tychonoff*) if for each closed $A \subseteq X$ and $b \in X \setminus A$ there exists an Urysohn function with respect to A and $\{b\}$ or equivalently if for each two disjoint closed subsets A and B of X such that at least one of these sets is compact there exists an Urysohn function μ with respect to A and B. A topological space is Tychonoff iff its topology is induced by a Hausdorff uniformity.

Recall that a topological space X is said to be normal if for each two disjoint closed subsets A and B of X there exists an Urysohn function $\mu : X \longrightarrow I$ vanishing on B and equal to 1 on A.

A compact space X is Tychonoff. Its topology determines a unique uniformity for X and vice versa. In absense of compactness this one-to-one correspondence fails. However frequently the topological space X under consideration enjoys an additional structure, e. g. X is metric or linear, and natural uniformities are given. In this cases we endow X with its natural uniformity without explicit mention.

1.2 Hausdorff topological vector spaces

The following basics of topological linear spaces are mainly taken from [Köt66]. For metrizable spaces we also refer to [Rol85] and the F-space sampler [KPR84].

Let E be a real vector space and τ a Hausdorff topology on E. E is said to be a *Hausdorff topological vector space* (short hand for HTVS) if its linear operations $(x, y) \mapsto x + y$ and $(\alpha, x) \mapsto \alpha x$, defined on $E \times E$ and $\mathbb{K} \times E$, respectively and mapping to E, are continuous.

A neighborhood U of the origin is said to be *circled* if $[-1, 1]U \subseteq U$. Each neighborhood U of the origin is *absorbant*, i. e. for any $x \in E$ there exists an $r > 0$ such that $sx \in U$ whenever $s \in [-r, r]$. The *Minkowski functional* $\varphi_U : E \longrightarrow [0, \infty)$ of U assigns each $x \in E$ the infimum over the above r. U is said to be *shrinkable* if its Minkowski functional is continuous. V. Klee [Kle60b] showed that every HTVS has a base of circled and shrinkable neighborhoods of the origin.

The topology τ of a HTVS is necessarily *translation invariant*, i. e. for any $x \in E$ a family (U) is a base of neighborhoods of the origin iff $(x + U)$ is a base of neighborhoods of x. Let (U) be a base of circled neighborhoods of the origin. Define $V_U := \{(x, y) \in E \times E\,;\, x - y \in U\}$, $U \in (U)$. Then (V_U) is a base of a translation invariant uniformity $\mathcal{V}(E)$ on E, i. e. for any $z \in E$ we have $(x, y) \in V_U$ iff $(x + z, y + z) \in V_U$. Since this correspondence is one-to-one the uniformity of a HTVS turns out to be unique. Let us therefore agree to use the terms *circled* and *convex* for vicinities $V \in \mathcal{V}(E)$, too, which means that these attributes hold for the corresponding neighborhoods $V(0) = \{x \in E\,;\, (x, 0) \in V\}$ of the origin 0.

S. Kakutani [Kak36] showed that for every metrizable HTVS E there exists a translation invariant metric which induces the topology τ of E. This metric can be induced by an *F-norm*, i. e. by a functional $\|\cdot\| : E \longrightarrow [0, \infty)$ which holds

(i) $\|\lambda x\| \leq \|x\|, \quad \lambda \in [-1, 1], \quad x \in E,$

(ii) $\lim_n \|\frac{1}{n}x\| = 0, \quad x \in E,$

(iii) $\|x + y\| \leq \|x\| + \|y\|, \quad x, y \in E.$

M. Eidelheit and S. Mazur [EM37] showed that it is possible to choose this F-norm *monotonic*, i. e. for any $x \in E$ the functional $\lambda \mapsto \|\lambda x\|$ is strictly increasing on $[0, \infty)$. An F-normed space which is complete is said to be an *F-space.*

A subset B of a HTVS is said to be *bounded* if for each sequence $(x_n)_n$ in B and each sequence $(t_n)_n$ in $\mathbb{R}$ which converges to 0 also $(t_n x_n)_n$ converges to 0. An F-normed space E is said to be *locally bounded* if E has a bounded neighborhood of the origin. An F-norm $\|\cdot\|$ is called a *p-norm* if it is p-homogeneous, i. e. $\|\lambda x\| = |\lambda|^p \|x\|$, $\lambda \in [-1, 1]$, $x \in E$. By T. Aoki [Aok42], S. Rolewicz [Rol57] a metrizable HTVS is locally bounded iff there exists a p-norm for E.

A HTVS E is said to be a *locally convex space* if E has a base of neighborhoods of the origin which are convex. A *Fréchet space* is a metrizable locally convex space which is complete. A *Banach space* is a locally bounded Fréchet space. By an *Euclidean space* we understand a Banach space with finite (algebraic) dimension. Note that for each $n \in \mathbb{N} \cup \{0\}$ there exists, modulo isomorphism, precisely one Euclidean space of dimension n which we denote by $\mathbb{R}^n$. We moreover use the term *Euclidean topology* to denote the topology of an Euclidean space if the dimension of that space is of minor interest.

1.3 Polyhedra

For the following definitions we follow mainly [Spa66] with slight supplement due to our purpose.

Let S be a non-empty set. A *simplicial complex* $\mathcal{S}$ is a system of non-empty finite subsets $\{\sigma\}$ of S, called *simplices*, such that if σ_1 is a simplex of $\mathcal{S}$ and $\sigma_2 \subseteq \sigma_1$ then σ_2 is a simplex of $\mathcal{S}$, too. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are simplicial complexes then $\mathcal{S}_2$ is said to be a *subcomplex* of $\mathcal{S}_1$ if $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

Despite some ambiguity for a finite set $\{x_0, \dots, x_n\}$ the simplicial complex Δ_n, consisting of all non-empty subsets of $\{x_0, \dots, x_n\}$, is called the *n-dimensional simplex*.

A simplicial complex $\mathcal{S}$ is said to be *finite* if its cardinality is finite. The *dimension* $\mathit{dim}\,\mathcal{S}$ of a simplicial complex $\mathcal{S}$ is one less than the least upper bound for the cardinalities of its containing simplices σ.

For all $0 \leq n \leq \mathit{dim}\,\mathcal{S}$ the *n-skeleton* $\mathcal{S}^n$ of the simplicial complex $\mathcal{S}$ is the subcomplex of $\mathcal{S}$ containing all simplices of $\mathcal{S}$ with cardinality at most $n+1$. The 0-skeleton $\mathcal{S}^0$ is also called the set of *vertices* of $\mathcal{S}$ and is identified with S.

For our purpose it is sufficient to consider only finite simplicial complexes. Let $\mathcal{S}$ be a finite simplicial complex and S its vertices. The real vector space E free generated by S has finite dimension, hence it has an uniquely determined Euclidean topology. Let $|\mathcal{S}|$ be the (topological) subspace of E given by $|\mathcal{S}| = \{x \in E\,;\, x \in \mathit{conv}\,\sigma,\, \sigma \in \mathcal{S}\}$.

A topological space P is said to be a *polyhedron* if there exists a finite simplicial complex $\mathcal{S}$ and a homeomorphism $h : |\mathcal{S}| \longrightarrow P$. The pair $(\mathcal{S}, h)$ is said to be a *triangulation* of the polyhedron P. For what follows we identify simplicial complexes and their associated polyhedra.

By a *convex polyhedron* we understand a polyhedron which is homeomorphic to a convex subset of an Euclidean space.

Any simplex Δ_n is a convex polyhedron. Note that in particular the one-point space $\{*\}$, cubes I^n, cones $\mathit{cone}\,P$ over convex polyhedra P and, more general, products $P_1 \times P_2$ and joins $P_1 \star P_2$ of convex polyhedra P_1, P_2 are convex polyhedra. We take a closer look at joins: suppose that P_1 and P_2 are convex polyhedra with triangulations $h_1 : |\mathcal{S}_1| \longrightarrow P_1$ and $h_2 : |\mathcal{S}_2| \longrightarrow P_2$. We can assume that P_1 and P_2 are convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Consider the embeddings $\iota_1 : \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1+m}$ and $\iota_2 : \mathbb{R}^m \hookrightarrow \mathbb{R}^{n+1+m}$ given by $\iota_1(x) := (x, 0, 0)$, $x \in \mathbb{R}^n$ and

$\iota_2(y) := (0, 1, y)$, $y \in \mathbb{R}^m$, respectively. We obtain a union of *segments*

$$P := \bigcup_{\substack{p_1 \in \iota_1(P_1) \\ p_2 \in \iota_2(P_2)}} [p_1, p_2] \qquad \text{where} \qquad [p_1, p_2] := \{(1-t)p_1 + tp_2\,;\, t \in I\} \tag{1.1}$$

which is a convex subset of R^{n+1+m} and homeomorphic to $P_1 \star P_2$. A triangulation of P is given by the simplicial complex $\mathcal{S} := \mathcal{S}_1 \uplus \mathcal{S}_2 \uplus \{\sigma_1 \uplus \sigma_2\,;\, \sigma_1 \in \mathcal{S}_1,\, \sigma_2 \in \mathcal{S}_2\}$ and the affine extension $h : |\mathcal{S}| \longrightarrow P$ of $h|_{|\mathcal{S}_1|} = h_1$ and $h|_{|\mathcal{S}_2|} = h_2$. Here $\uplus$ denotes the disjoint union.

We frequently make use of notation (1.1) when dealing with joins of polyhedra and omit explicit mentioning of the embeddings ι_1 and ι_2 in what follows.

Observe that if we make use of a convex polyhedron P it makes sense to consider convex combinations $tp_1 + (1-t)p_2$, $t \in I$ of $p_1, p_2 \in P$.

Remark 1.1. Even so we will not make explicit use of realizations of simplicial complexes we remark that every finite simplicial complex $|\mathcal{S}|$ admits a *realization* in some Euclidean space $\mathbb{R}^n$, i. e. there exists an affine embedding of $|\mathcal{S}|$ into $\mathbb{R}^n$. Indeed, such an embedding can be constructed by the affine extension of an identification of the vertices $\mathcal{S}^0$ with pairwise disjoint elements of $\mathbb{R}^{2\, dim\, \mathcal{S}+1}$ such that any $(2(dim\, \mathcal{S}+1))$ of the vertices are in general position, i. e. they are affinely independent.

Observe that for a convex polyhedron, say $|\mathcal{S}|$, in general the above realization of $|\mathcal{S}|$ need not be a convex subset of $\mathbb{R}^n$.

Let X be a non-empty topological space and $\mathcal{O}$ an open covering of X. The *nerve* of $\mathcal{O}$ is the simplicial complex $\mathcal{S}$ given by the system of all non-empty finite subsets of $\mathcal{O}$ which have non-empty intersection. The *covering dimension* (frequently called *topological dimension* or *dimension*) of X is the smallest number $n \in \mathbb{N} \cup \{0\}$ such that every open covering of X admits an open refinement whose nerve is at most n-dimensional as a simplicial complex. A topological space X is said to be *finite dimensional* if its covering dimension is finite. Recall that the covering dimension is non-decreasing with respect to set-inclusion and that for the Euclidean spaces $\mathbb{R}^n$ covering- and algebraic dimension coincide.

Let $\mathcal{O}$ be a finite covering of a compact space X, $\mathcal{S}$ its nerve and $\{\lambda_O\,;\, O \in \mathcal{O}\}$ a partition of unity subordinated to $\mathcal{O}$. Then the *barycentric function* $b : X \longrightarrow \mathcal{S}$ (with respect to $\mathcal{O}$ and $\{\lambda_O\,;\, O \in \mathcal{O}\}$) is defined by

$$b(x) := \sum_{O \in \mathcal{O}} \lambda_O(x)\{O\}, \quad x \in X.$$

Observe that b maps x to that subcomplex of $\mathcal{S}$ which corresponds to the nerve of $\{O \in \mathcal{O}\,;\, x \in O\}$. In particular if x belongs to one and only one O of $\mathcal{O}$ then $b(x) = \{O\}$.

1.4 Maps, functions and selections

For basic facts on set-valued maps (short hand for maps) we refer to [Ber59] and [AF90].

Let A and B be sets. By 2^B we denote the set of all non-empty subsets of B. A *map* $F : A \longrightarrow 2^B$ assigns to each $a \in A$ the non-empty set $F(a) \subseteq B$. In case of singletons $F(a) = \{b\}$ we call F a *function* and identify $\{b\}$ and b. To emphasize the singleton-valued character of functions we use lower characters $F = f$. In literature maps frequently are called *set-valued maps*, *correspondences, carriers* or *multis*. The *graph* of F is the subset $\mathcal{G}(F) := \{(a, b) \in A \times B\,;\, b \in F(a)\}$ of $A \times B$.

A function f is said to be a *selection* of a map F if $\mathcal{G}(f) \subseteq \mathcal{G}(F)$. We often use the notation

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "f"'] & & B \\ & B \arrow[ur, hook, "id"'] & \end{tikzcd} \tag{1.2}$$

for selections. The selection f is said to be a *continuous selection* if A and B are topological spaces and f is continuous.

Let $\tilde{A} \subseteq A$. The *restriction* $F|_{\tilde{A}} : \tilde{A} \longrightarrow 2^B$ of a map $F : A \longrightarrow 2^B$ is given by means of its graph $\mathcal{G}(F|_{\tilde{A}}) = \mathcal{G}(F) \cap (\tilde{A} \times B)$. A map $\tilde{F} : A \longrightarrow 2^B$ is said to be an *extension* of a map $F : \tilde{A} \longrightarrow 2^B$ if F is a restriction of $\tilde{F}$.

Let $F : A \longrightarrow 2^B$ be a map and $\tilde{A} \subseteq A$, $\tilde{B} \subseteq B$. The set $F(\tilde{A}) := \cup_{a \in \tilde{A}} F(a)$ is the *image* of $\tilde{A}$ under F, $F^{=1}(\tilde{B}) := \{a \in A\,;\, F(a) \subseteq \tilde{B}\}$ and $F^{-1}(\tilde{B}) := \{a \in A\,;\, F(a) \cap \tilde{B} \neq \emptyset\}$ are the *small-* and *large preimage* of $\tilde{B}$ under F, respectively. In literature small and large preimages are frequently called *core* and *inverse*. To any map $F : A \longrightarrow 2^B$ one assigns its *inverse* $F^{-1} : F(A) \longrightarrow 2^A$ by changing ordinate and abscissa, i. e. $F^{-1}(b) := F^{-1}(\{b\})$, $b \in F(A)$. For maps $F : A \longrightarrow 2^B$ and $G : B \longrightarrow 2^C$ their *concatenation* $(GF) : A \longrightarrow 2^C$ is given by $(GF)(a) := \cup_{b \in F(a)} G(b)$, $a \in A$. Furthermore we use the notation $(F \times G) : A \times B \longrightarrow 2^{C \times D}$ for the map given by $F(a) \times G(b)$, $a \in A$, $b \in B$ where $F : A \longrightarrow 2^C$ and $G : B \longrightarrow 2^D$ are given maps. If $C = D$ the definitions of $F \cup G$ and $F \cap G$ are obvious. If $A = B$ and $F = f$, $G = g$ are single-valued let $(f, g) : A \longrightarrow C \times D$ denote the function given by $(f(a), g(a))$, $a \in A$.

If A is a subset of B a point $x \in A$ is said to be a *fixed point* of the map $F : A \longrightarrow 2^B$ if $x \in F(x)$. By $Fix(F)$ we denote the set of all fixed points of F.

If A and B are topological spaces, a map F is said to be *closed* if its graph is closed.

F is said to be *upper semicontinuous* at $x \in A$ if $F^{=1}(O)$ is open provided that $O \subseteq B$ is an open neighborhood of $F(x)$. F is said to be *lower semicontinuous* at $x \in A$ if $F^{-1}(O)$ is a neighborhood of x provided that $O \subseteq B$ is open and $F(x) \cap O \neq \emptyset$.

Let $\tilde{A} \subseteq A$. F is said to be upper (lower) semicontinuous on $\tilde{A}$ if it is upper (lower) semicontinuous at each point of $\tilde{A}$.

A map $F : A \longrightarrow 2^B$ is said to be *compact* if its image $F(A)$ is relatively compact. It is well-known that every closed compact map F is upper semicontinuous. In case of singleton-valued maps, i. e. functions $F = f$, the notions of upper- and lower semicontinuity coincide with continuity of functions.

Observe that for a Hausdorff uniform space B, closed $A \subseteq B$ and closed (compact) $F : A \longrightarrow 2$ the set of fixed points $Fix(F)$ is closed (compact).

Of special interest are maps $x \mapsto V(x)$ where V is a vicinity of a uniform space X. Evidently V is identical with the graph of this map. If there is no misinterpretation possible we use the symbol V to denote the map as well as its graph. With this notation graphs of closed maps $F : X \longrightarrow 2^Y$ between uniform spaces $(X, \mathcal{V}(X))$ and $(Y, \mathcal{V}(Y))$ enjoy the representation

$$\mathcal{G}(F) = \bigcap_{\substack{U \in \mathcal{V}(X) \\ V \in \mathcal{V}(Y)}} \mathcal{G}(VFU) \tag{1.3}$$

which is fundamental for all of our purposes.

We will make use of the following selection theorem for a map $F : X \longrightarrow 2^E$. It is due to E. Michael [Mic56, Theorem 3.2”] for Banach spaces E. A more general version can be found in [BP75, Chapter II, Theorem 7.1, Corollary 7.5] where E is a locally convex space and F a complete carrier, i. e. the images $F(x)$ are complete spaces.

Theorem 1.1 (Michael selection theorem). *Let X be a paracompact space, $\hat{X} \subseteq X$ closed and E a Fréchet space. Let $F : X \longrightarrow 2^E$ be lower semicontinuous with closed convex values. Then every continuous selection f' of $F|_{\hat{X}}$ can be extended to a continuous selection f of F.*

Proof. Consider the map $F' : X \longrightarrow 2^E$ given by $F'(x) := f'(x)$ if $x \in \hat{X}$ and $F'(x) := F(x)$ if $x \in X \setminus \hat{X}$. Then F' fulfills the same assumptions as F and the existence of the continuous selection follows from [BP75, Chapter II, Theorem 7.1, Corollary 7.5]. $\square$

Remark 1.2. We remark that closed convex-valued maps between compact convex subsets of Fréchet spaces does not necessarily admit a continuous selection.

To see this we modify an example given by E. Michael [Mic56, Example 6.1].

Let $X = I$ and $E = I^2$ be the unit interval and the unit square, respectively. Consider the map $F : X \longrightarrow 2^E$ defined by

$$F(x) := \begin{cases} conv \left\{ \left(t, \left|cos \frac{1}{t}\right|\right) ; x - x^3 \le t \le x \right\}, & x \neq 0 \\ \{0\} \times I, & x = 0 \end{cases}$$

which is closed and convex-valued. F does not have a continuous selection. Indeed, if there would exist one, say f, the image of f would define an arc from some point

$(0, y_1)$ to some point $(1, y_2)$ contained in the image of F. But since $x - x^3 \leq t \leq x$ iff $\frac{1}{x} \leq \frac{1}{t} \leq \frac{1}{x} + \frac{x}{1-x^2}$, $0 < x < 1$ the sets $F(x)$ tend to singleton $(x, |cos\frac{1}{x}|)$ for x becoming small. Thus the image of F behaves like the well-known counterexample for a connected but not arcwise connected space.

Note that, by Theorem 1.1, F cannot be lower semicontinuous. In fact F is not lower semicontinuous in 0.

However Lemma 1.1 below shows that F admits selections in an approximative sense.

The following lemma is due to S. Kakutani [Kak41] for Euclidean spaces E and to H. F. Bohnenblust, S. Karlin [B50], K. Fan [Fan52] for Banach spaces and locally convex spaces, respectively. See also [DG82, (11.2) Lemma] for metric X.

Lemma 1.1. *Let X be a paracompact space, E a locally convex space and $\hat{X} = \cup_{i=1}^n \hat{X}_i$ a pairwise disjoint union of compacta $\hat{X}_i \subseteq X$. Let $F : X \longrightarrow 2^E$ be an upper semicontinuous map with closed convex values. Let U and V be vicinities in X and E, respectively. Then every selection f' of $F|_{\hat{X}}$ which is constant on each $\hat{X}_i$ can be extended to a selection f of VFU.*

Proof. If $\hat{X} = \emptyset$ we are concerned with the already mentioned classical selection result. If $\hat{X} \neq \emptyset$ we prove by induction with respect to n and use the classical result as the starting point of the induction. Let f', $\cup_{i=1}^n \hat{X}_i$ and V, U be given as above.

Since E is locally convex there exists a convex vicinity $\tilde{V}$ in E such that $\tilde{V}\tilde{V} \subseteq V$. Since F is uniformly upper semicontinuous on $\hat{X}$ there exists a symmetric vicinity $\tilde{U}$ in X such that $\tilde{U} \subseteq U$ and $(F\tilde{U})(\hat{X}_n) \subseteq (\tilde{V}F)(\hat{X}_n)$. Thus

$$(\tilde{V}F\tilde{U})(\hat{X}_n) \subseteq (\tilde{V}\tilde{V}F)(\hat{X}_n). \tag{1.4}$$

X, as a paracompact space, is completely regular and we can assume that $\tilde{U}(\hat{X}_n)$ is open and disjoint to $\cup_{i=1}^{n-1} \hat{X}_i$. By hypothesis there exists a selection $\tilde{f} : X \longrightarrow E$ of $\tilde{V}F\tilde{U}$ extending $f'|_{\cup_{i=1}^{n-1} \hat{X}_i}$. Choose an Urysohn function $\mu : X \longrightarrow I$ equal to 1 on $X \setminus \tilde{U}(\hat{X}_n)$ and vanishing on $\hat{X}_n$.

Define $f : X \longrightarrow E$ by

$$f(x) := \mu(x)\tilde{f}(x) + (1 - \mu(x))f'(\hat{X}_n), \quad x \in X.$$

f is continuous and $f|_{\hat{X}} = f'$. Since $f = \tilde{f}$ on $X \setminus \tilde{U}(\hat{X}_n)$ and $\mathcal{G}(\tilde{f}) \subseteq \mathcal{G}(\tilde{V}F\tilde{U}) \subseteq \mathcal{G}(VFU)$ it remains to show $\mathcal{G}(f|_{\tilde{U}(\hat{X}_n)}) \subseteq \mathcal{G}(VFU)$.

By construction $\mathcal{G}\big(f|_{\tilde{U}(\hat{X}_n)}\big) \subseteq \mathcal{G}\big(conv\{\tilde{f}, f'(\hat{X}_n)\}|_{\tilde{U}(\hat{X}_n)}\big)$ and by hypothesis the latter set is contained in $\mathcal{G}\big(conv\{\tilde{V}F\tilde{U}, f'(\hat{X}_n)\}|_{\tilde{U}(\hat{X}_n)}\big)$. Since $f'(\hat{X}_n) \in F(x)$, $x \in \hat{X}_n$ and $\tilde{U}$ is symmetric we infer $f'(\tilde{U}(\hat{X}_n)) \subseteq F\tilde{U}(x)$, $x \in \tilde{U}(\hat{X}_n)$. Thus, $\tilde{V}$ being convex, we infer $\mathcal{G}\big(f|_{\tilde{U}(\hat{X}_n)}\big) \subseteq \mathcal{G}\big(\tilde{V}F\tilde{U}|_{\tilde{U}(\hat{X}_n)}\big)$ and the latter set is contained in $\mathcal{G}(VFU)$, hence $\mathcal{G}\big(f|_{\tilde{U}(\hat{X}_n)}\big) \subseteq \mathcal{G}(VFU)$. □

One can show that Lemma 1.1 also holds if E is replaced by a convex subset K of E. Moreover f maps to a convex polyhedron provided X is a polyhedron. In fact, this is well-known for $\hat{X} = \emptyset$ and, in view of our proof, this holds for general $\hat{X}$, too.

1.5 Extension, factorization and approximation of functions

The classical Tietze extension theorem, see e. g. [Kel55], states for a closed subset A of a normal space X that every continuous function $f : A \longrightarrow I$ admits a continuous extension $\hat{f} : X \longrightarrow I$. For completely regular spaces this reads as follows.

Theorem 1.2 (Tietze extension theorem). *Let X be a completely regular space and $A \subseteq X$ compact. Let $f : A \longrightarrow I^n$ be continuous. Then f admits a continuous extension $\hat{f} : X \longrightarrow I^n$.*

The following extension theorem goes back to J. Dugungji [Dug51]. A proof can e. g. be found in [DG82, (10.4) Theorem].

Theorem 1.3 (Dugundji extension theorem). *Let X be a metric and E a locally convex space. Let $A \subseteq X$ be closed and $f : A \longrightarrow E$ continuous. Then f admits a continuous extension $\hat{f} : X \longrightarrow E$ such that $\hat{f}(X) \subseteq \overline{conv} f(A)$.*

In its original form the following approximation result goes back to Tychonoff [Tyc35], see also [DG82] or [Rie76] for normed spaces. We need a small adaption which reads as follows

Theorem 1.4 (Schauder projection). *Let E be a locally convex space, $K \subseteq E$ non-empty and compact and $K' \subseteq K$ finite. Let V be a vicinity in E. Then there exists a convex polyhedron $P \subseteq K$ and a continuous function $s : K \longrightarrow P$ such that $(id, s)(K) \subseteq V$ and $s(x) = x$ for $x \in K'$.*

Proof. We prove by induction with respect to the cardinality of K' and use the classical result as the starting point of the induction.

Assume $K' = \{x_1, \ldots, x_n\}$. By hypothesis there exists a convex polyhedron P and a continuous $s : K \longrightarrow P$ such that $s(x_i) = x_i$, $i = 1, \ldots, n-1$ and $(id, s)(K) \subseteq U$ for some starshaped open vicinity U with $UU \subseteq V$. Let $\mu : K \longrightarrow I$ be an Urysohn function equal to 1 on $X \setminus U(x_n)$ and vanishing on x_n. The function $s' : K \longrightarrow conv(P \cup \{x_n\})$ given by $s'(x) := \mu(x)s(x) + (1-\mu(x))x_n$ is continuous and maps to the polyhedron $conv\,(P \cup \{x_n\})$. Since $x - s'(x) = \mu(x)(x - s(x)) + (1 - \mu(x))(x - x_n) \in U(0) + U(0) \subseteq V(0)$ for any $x \in K$ the function s' has the desired properties with respect to K'. $\square$

We end this section with a fundamental embedding result. See [Tor72, Addendum] and [EF78, Satz A.2.1] for the proof of

Theorem 1.5 (Arens-Eells). *Every metric space X can be embedded as a closed subspace into a normed linear space. If X is in addition compact it can be embedded as a closed subspace into the Hilbert cube I^∞.*

1.6 Absolute (neighborhood) retracts and (neighborhood) extension spaces

This chapter follows mainly [Han52], [Bor67], [Hu65] and [vM89]. See also [DG82] and [EF78] for formulations more customized to fixed point-theory.

Definition 1.1. Let X be a topological space. A non-empty closed subset A of X is said to be a *retract* of X if the identity $id : A \longrightarrow A$ admits a continuous extension $r : X \longrightarrow A$.

A is said to be a *neighborhood retract* of X if there exists a neighborhood U of A in X such that A is a retract of U.

Obviously every retract of a space is also a neighborhood retract of that space.

E. g. the Dugundji extension theorem, Theorem 1.3, shows that each non-empty closed convex subset of a metrizable locally convex space is a retract of that space.

A function $\phi : X \hookrightarrow Y$ is said to be a *topological embedding* if $\phi : X \longrightarrow \phi(X)$ is a homeomorphism.

Definition 1.2. Let $\mathcal{X}$ be a class of topological spaces. A space $X \in \mathcal{X}$ is said to be an *absolute retract* for the class $\mathcal{X}$ (short hand for X *is* $AR(\mathcal{X})$) if for every $Y \in \mathcal{X}$ and every topological embedding $\phi : X \hookrightarrow Y$ with closed image $\phi(X)$, $\phi(X)$ is a retract of Y.

In a similar manner the term *absolute neighborhood retract* for the class $\mathcal{X}$ (short hand for X *is* $ANR(\mathcal{X})$) is defined.

The cubes I^n are $AR(normal)$ by the Tietze extension theorem for normal spaces. Every closed convex subset of a metrizable locally convex space is $AR(metric)$ by the Dugundji extension theorem. Polyhedra and, more general, compact manifolds are $ANR(metric)$.

An extensive examination of the $A(N)R$-property with respect to a wide variety of subclasses of the class of completely regular spaces give [Han52], [Bor67] and [Hu65]. For our purpose it is sufficient to consider $\mathcal{X} \in \{$*metric, compact metric,* *polyhedra, finite dimensional metric*$\}$. Recall the inclusions

$$\begin{tikzcd}
 & \textit{finite dimensional metric} \arrow[dr] & \\
\textit{polyhedra} \arrow[ur] \arrow[dr] & & \textit{metric} \\
 & \textit{compact metric} \arrow[ur] &
\end{tikzcd}$$

There exists a variety of characterizations of the $A(N)R$-property for the classes of all metric and all compact metric spaces, respectively. Due to our purpose we take the following ones from [BP75, II, Proposition 5.5] and [EF78, Satz A.2.8, Satz A.2.10], respectively. Their proofs relie essentially on the Arens-Eells embedding theorem, see Theorem 1.5. For a proof we refer to [Tor72] or [DG82, (B.13) Theorem] and [EF78, Satz A.2.8], respectively.

Proposition 1.1. *Let X be a metric space. Then the following statements are equivalent:*

(i) *X is $A(N)R(metric)$,*

(ii) *X can be embedded as a (neighborhood) retract into a normed linear space,*

(iii) *for every metric space Y and closed $A \subseteq Y$ every continuous function $f : A \longrightarrow X$ admits a continuous extension over (a neighborhood of A in) Y.*

Proposition 1.2. *Let X be a compact metric space. Then the following statements are equivalent:*

(i^c) *X is $A(N)R(compact\ metric)$,*

(ii^c) *X can be embedded as a (neighborhood) retract into the Hilbert cube I^∞,*

(iii^c) *for every compact metric space Y and closed $A \subseteq Y$ every continuous function $f : A \longrightarrow X$ admits a continuous extension over (a neighborhood of A in) Y.*

The following statements are essential to establish a fixed point-theory for $ANR(compact\ metric)$.

Definition 1.3. Let X be a topological space and $\mathcal{O}$ an open cover of X. A topological space Y is said to be an *$\mathcal{O}$-domination* of X if there exists a factorization

$$\begin{tikzcd}
X \arrow[rr, "gf"] \arrow[dr, "f"'] & & X \\
 & Y \arrow[ur, "g"'] &
\end{tikzcd} \tag{1.5}$$

such that gf is *$\mathcal{O}$-homotopic* to the identity, i. e. there exists a homotopy $h^t : X \longrightarrow X$ such that $h^0 = id$, $h^1 = gf$ and every arc $t \mapsto h^t(x)$, $x \in X$ is contained in a member of $\mathcal{O}$.

Theorem 1.6. *X is ANR(compact metric) iff X is compact metric and for each open covering $\mathcal{O}$ of X there exists a polyhedron P which $\mathcal{O}$-dominates X.*

Theorem 1.7. *X is AR(compact metric) iff X is ANR(compact metric) and contractible.*

Characterizations (iii) and (iii^c) of Propositions 1.1 and 1.2, respectively, give rise to

Definition 1.4. Let $\mathcal{X}$ be a class of topological spaces. A topological space X is said to be an *absolute extensor* for the class $\mathcal{X}$ (short hand for X *is* $AE(\mathcal{X})$) if for every space $Y \in \mathcal{X}$ and closed $A \subseteq Y$ every continuous function $f : A \longrightarrow X$ admits a continuous extension over Y.

A topological space X is said to be an *absolute neighborhood extensor* for the class $\mathcal{X}$ (short hand for X *is* $ANE(\mathcal{X})$) if for every space $Y \in \mathcal{X}$ and closed $A \subseteq Y$ every continuous function $f : A \longrightarrow X$ admits a continuous extension over a neighborhood of A.

By Propositions 1.1 and 1.2 the difference between absolute (neighborhood) retracts and absolute (neighborhood) extensors for a class of spaces $\mathcal{X}$ turns out to be that the latter ones are not necessarily a elements of $\mathcal{X}$. E. g. the Dugundji extension theorem does not suppose metrizabiliy of E, i. e. each closed convex subset of a locally convex space E is AE(*metric*).

Closely related to Theorem 1.7 is

Theorem 1.8. *Let $\mathcal{X}$ be a subclass of the class of all normal spaces. Then every contractible $ANE(\mathcal{X})$ is $AE(\mathcal{X})$.*

We end this part with some customized extensions for later use.

Lemma 1.2. *Let X be a topological space which is locally contractible at a point $x_0 \in X$. Then for any neighborhood O of x_0 there exists a neighborhood $Q \subseteq O$ of x_0 such that for every topological space W and every continuous function $f : W \longrightarrow Q$ there exists a continuous extension $\hat{f} : \mathit{cone}\, W \longrightarrow O$ of f such that $\hat{f}(\star) = x_0$ where $\star$ is the tip of cone W.*

Moreover, if W is metrizable and W' a non-empty closed subset of W where $f = x_0$ holds then an extension $\hat{f}$ exists such that in addition $\hat{f} = x_0$ on cone W'.

Proof. Fix a neighborhood O of x_0. Then there exists a neighborhood Q of x_0 and a homotopy $h : [0,1] \times Q \longrightarrow O$ such that $h(0,\cdot) = x_0$ and $h(1,\cdot) = \iota : Q \hookrightarrow O$. Let $*$ denote the tip of the cone over W. The function $\hat{f} : \mathit{cone}\, W \longrightarrow O$ given by

$$\hat{f}(tw + (1-t)*) := h(t, f(w)), \quad w \in W, \quad t \in [0,1] \tag{1.6}$$

is well-defined, continuous, has image contained in O and extends f.

If W is metrizable let d be a metric for W and replace (1.6) by

$$\hat{f}(tw + (1-t)*) := h(td(w, W'), f(w)), \quad w \in W, \quad t \in [0,1],$$

where $d(w, W') := \inf\{d(w, w')\,;\, w' \in W'\}$. □

Lemma 1.3. *Let X be a contractible space, $\mathcal{S}$ a finite simplicial complex and $\tilde{\mathcal{S}}$ a subcomplex of $\mathcal{S}$. Then every continuous function $f : \tilde{\mathcal{S}} \longrightarrow X$ admits a continuous extension $\hat{f} : \mathcal{S} \longrightarrow X$.*

Proof. If $\mathcal{S}$ is the empty simplicial complex there is nothing to do. Otherwise we construct $\hat{f}$ successively with respect to the dimensions of the skeletons of $\mathcal{S} \setminus \tilde{\mathcal{S}}$.

Let $\hat{f} := f$ on $\tilde{\mathcal{S}}$.

If there is no $n \in \mathbb{N} \cup \{0\}$ such that $\mathcal{S}^n \setminus \tilde{\mathcal{S}}^n \neq \emptyset$ no extension is needed. Otherwise there exists a maximal $n \in \mathbb{N}$ such that $\hat{f} = f$ on $\mathcal{S}^{n-1}$ (let $\mathcal{S}^{-1}$ be the empty set) and there is at least one simplex $\sigma \in \mathcal{S}^n \setminus \tilde{\mathcal{S}}^n$.

Since X is contractible there exists a homotopy $h : I \times X \longrightarrow X$ such that $h(1, \cdot) = id$ and $h(0, \cdot) = * \in X$. We can identify σ with some Euclidean ball $\{x\,;\, |x| \leq 1\}$ such that the boundary $\partial_{(\mathcal{S}^{n-1} \cup \sigma)}\sigma$ of σ is identified with $\{x\,;\, |x| = 1\}$. Since $\partial_{(\mathcal{S}^{n-1} \cup \sigma)}\sigma \subseteq \mathcal{S}^{n-1}$ we can define $\hat{f} : \mathcal{S}^{n-1} \cup \sigma \longrightarrow X$ by

$$\hat{f}(x) := \begin{cases} \hat{f}(x), & x \in \mathcal{S}^{n-1} \\ h\left(|x|, \hat{f}\left(\frac{x}{|x|}\right)\right), & x \in \sigma. \end{cases}$$

Then $\hat{f}$ is continuously extended to $\mathcal{S}^{n-1} \cup \sigma$.

We continue this construction until $\mathcal{S}^n \setminus \tilde{\mathcal{S}}^n = \emptyset$ and obtain a continuous extension $\hat{f} : \mathcal{S}^n \longrightarrow X$. Now proceed with the skeleton $\mathcal{S}^{n+1}$.

Since the simplicial complex $\mathcal{S}$ is finite this procedure terminates. □

A sophisticated use of local contractibility leads to a generalization of Lemma 1.2 which provides extensions of continuous functions given on closed subsets of finite dimensional metric spaces. More precisely, we have the following proposition whose proof can be found in [Bor67, Chapter III (9.1) Theorem.].

Proposition 1.3. *Let X be a locally contractible metrizable space. Then for any $x_0 \in X$ and $V \in \mathcal{V}(X)$ there exists $V' \in \mathcal{V}(X)$ such that for every finite dimensional metric space Y, closed $W \subseteq Y$ and continuous $f : W \longrightarrow V'(x_0)$ there exists a continuous extension $\hat{f} : Y \longrightarrow V(x_0)$ of f.*

The statement of the above Proposition is equivalent to X *is* ANE(finite dimensional metric), see [Bor67, Chapter III (9.1) Theorem. (2)]. Hence we infer straightforwardly from Theorem 1.8

Lemma 1.4. *Every locally contractible metric space X is ANE(finite dimensional metric).*

If X is in addition contractible then X is AE(finite dimensional metric).

1.7 Some classical fixed point theorems

In general we refer to [DG82]. A Compendium in character is the monograph [vdW63]. More topological in nature and with main emphasis to set-valued maps is [Gór99].

Definition 1.5. A topological space X is said to be a *fixed point space* if each continuous function $f : X \longrightarrow X$ has a fixed point.

We start with some well-known fixed point theorems. Proofs can e. g. be found in [DG82].

Theorem 1.9 (Browder fixed point theorem). *Let $f : I^n \longrightarrow \mathbb{R}^n$ be continuous and suppose that f is odd on the boundary of I^n, i. e. $f(x) = -f(-x)$, $x \in S^{n-1}$. Then f has a fixed point.*

Theorem 1.10 (Brouwer fixed point theorem). *Let P be a convex polyhedron and $f : P \longrightarrow P$ a continuous function. Then f has a fixed point.*

Theorem 1.11 (Kakutani fixed point theorem). *Let K be a closed convex subset of a locally convex space and $F : K \longrightarrow 2^K$ closed, compact and convex-valued. Then F has a fixed point.*

Theorem 1.12 (Banach fixed point theorem). *Let (M, d) be a complete metric space and $f : M \longrightarrow M$ contractive, i. e. Lipschitz continuous with Lipschitz constant less than 1. Then f has precisely one fixed point.*

Standard references for the following homological terms are [ES52], [Spa66]. More related to fixed point theory are [Gra01], [Gór76], [Gór99] and [EF78]. We follow the notation of the latter one.

We use simplicial homology over the field of rationals. For any polyhedron P its homology groups H_q are vector spaces and the graded space (H_q) is of finite type, i. e. the H_q are finite dimensional and eventually trivial. Thus for any continuous $f : P \longrightarrow P$ and assigned homomorphisms $f_*^q : H_q \longrightarrow H_q$, $q = 0, 1, \ldots$ the *Lefschetz number*

$$\Lambda(f) = \sum_{q=0}^{\infty} (-1)^q tr(f_*^q),$$

where tr denotes the trace, is well-defined and finite.

The classical Lefschetz-Hopf fixed point theorem states that every continuous self-function f of a polyhedron P has a fixed point provided $\Lambda(f) \neq 0$.

By Theorem 1.6 and by means of invariance of homology with respect to homotopic equivalence this result generalizes to ANR(*compact metric*):

Theorem 1.13 (Lefschetz fixed point theorem). *Let X be ANR(compact metric) and $f : X \longrightarrow X$ continuous such that $\Lambda(f) \neq 0$. Then f has a fixed point.*

By Theorem 1.7 every AR(compact metric) X is contractible and hence *acyclic*, i. e. X has the homology of the one-point space. Therefore $\Lambda(f) = 1$ for any continuous self-function f of X and we obtain

Corollary 1.1. *Every AR(compact metric) is a fixed point space.*

The corresponding set-valued version reads as

Theorem 1.14 (Eilenberg Montgomery fixed point theorem). *Let X be AR(compact metric) and $F : X \longrightarrow X$ closed with acyclic sets as values. Then F has a fixed point.*

2 The general approximation concept

In what follows $(X, \mathcal{V}(X))$ is a uniform space and $\mathcal{V}(X)$ a base for the uniformity of X consisting of symmetric vicinities. Let us agree that $\mathcal{A}(X)$ denotes the system of all finite subsets of X and $\Gamma(X) := \mathcal{A}(X) \times \mathcal{V}(X)$.

2.1 Approximation of the diagonal

Consider a closed map $F : X \longrightarrow 2^X$ and fix $(X', V) = \gamma \in \Gamma(X)$. By a *triangle*

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \gamma & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd} \tag{2.1}$$

we understand the following properties of φ, ψ and P:

(i) P is a convex polyhedron and $\varphi : X \longrightarrow P$, $\psi : P \longrightarrow X$ are continuous functions,

(ii) $Fix(\psi\varphi) \subseteq Fix(VFV)$ $\quad$ (V*-proximity*),

(iii) $Fix(F) \cap X' \subseteq Fix(\psi\varphi)$ $\quad$ (*preciseness on* X').

If evident what map F is under consideration the notions V*-proximity* and *preciseness on* X' are given for further reference to conditions (ii) and (iii), respectively. Short hand for we will make use of the slogans *proximity* and *preciseness*, whenever it is obvious what V and X', respectively, are under consideration or whenever specific knowledge of V and X', respectively, is unimportant.

Observe that for fixed point-free F preciseness is redundant for the triangle (2.1). On the other hand the fixed point property of VFV is necessary for V-proximity, hence for the triangle (2.1): Indeed, by the Brouwer fixed point theorem $(\varphi\psi) : P \longrightarrow P$ has at least one fixed point p. Thus $\psi(p)$ is a fixed point of VFV.

In terms of graphs V-proximity and preciseness on X' read as

$$\mathcal{G}(\psi\varphi) \cap \Delta(X) \subseteq \mathcal{G}(VFV) \quad \text{and} \quad \mathcal{G}(F) \cap \Delta(X') \subseteq \mathcal{G}(\psi\varphi), \tag{2.2}$$

respectively. Observe the difference between (2.1) and the 'empty' triangle in (1.2) which denotes a selection.

Let

$$(F,\gamma) := \{(\psi,\varphi)\,;\, \psi \text{ and } \varphi \text{ constitute a triangle (2.1)}\} \tag{2.3}$$

and associate to each pair $(\psi,\varphi) \in (F,\gamma)$ its convex polyhedron P. Furthermore identify $(X', X\times X) = X'$ and $(\emptyset, V) = V$ for any $(X',V) \in \Gamma(X)$, i. e. we use the notations (F,X') and (F,V), too, provided there is no misunderstanding possible.

We formulate now our general idea of approximating the fixed point-set of an arbitrary map $F : X \longrightarrow 2^X$. Note that $(F,U) \subseteq (F,V)$ whenever $U \in \mathcal{V}(X)$, $V \in \mathcal{V}(X)$ such that $U \subseteq V$. Thus the following definition is independent of the chosen base $\mathcal{V}(X)$ of the underlying uniform space X.

Definition 2.1. A closed map $F : X \longrightarrow 2^X$ is said to be *approximable* if for each $\gamma \in \Gamma(X)$ there exists a triangle (2.1), i. e. the set (F,γ) is non-empty.

In general approximability is closely related to the uniformity on X.

Remark 2.1. Consider the space $X = [0,\infty)$ with its Euclidean topology. We consider three different uniformities on X. Let $\mathcal{V}_0(X)$ be the natural uniformity induced by the Euclidean metric on X and $\mathcal{V}_1(X)$, $\mathcal{V}_2(X)$ be the trace-uniformities if one considers X as subspace of its Alexandroff and Stone-Čech compactification, respectively. Observe that $\mathcal{V}_1(X) \subseteq \mathcal{V}_0(X) \subseteq \mathcal{V}_2(X)$ and that these inclusions are proper.

Consider the function $f : X \longrightarrow X$ given by $f(x) := x + \frac{1}{1+x}$, $x \in [0,\infty)$. We claim f is approximable with respect to $\mathcal{V}_0(X)$ and $\mathcal{V}_1(X)$ but fails to be approximable with respect to $\mathcal{V}_2(X)$.

Note first that f is fixed point-free. Thus approximability is equivalent to V-proximity for all vicinities V of the uniformity under consideration.

First fix $V \in \mathcal{V}_0(X)$. There exists an $n \in \mathbb{N}$ such that $n \in Fix(VfV)$. Thus

$$\begin{tikzcd}
{[0,\infty)} \arrow[rr, "f"] \arrow[dr, "n"'] & V & {[0,\infty)} \\
& \{n\} \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

for such an n. Hence (f,V) is non-empty and f approximable with respect to the uniformity $\mathcal{V}_0(X)$ and, therefore, also with respect to the uniformity $\mathcal{V}_1(X)$.

If X is embedded in its Stone-Čech compactification the function $x \mapsto x^2$ is uniformly continuous on $[0,\infty)$. Thus for all $r > 0$

$$V_r := \big\{(x,y)\,;\, \max\{|x-y|, |x^2-y^2|\} \le r\big\}$$

is a vicinity in $[0,\infty)$. Since f is non-expansive, i. e. $|f(x)-f(y)| \le |x-y|$, $x,y \in X$, and $V_rV_r \subseteq V_{2r}$ we infer $\mathcal{G}(V_rfV_r) \subseteq \mathcal{G}(V_{2r}f)$. Hence

$$Fix(V_rfV_r) \subseteq \left\{ x \in [0,\infty)\,;\, \max\left\{ x - \left(x + \frac{1}{1+x}\right), x^2 - \left(x + \frac{1}{1+x}\right)^2 \right\} \le 2r \right\}.$$

The above inequality for x is equivalent to $\max\{1+x, 2x^3-x^2-1\} \le 2r(1+x)^2$, hence it fails for all $0 < r \le r_0$ provided r_0 is sufficiently small.

Thus $V_r f V_r$ is fixed point-free, hence f is not V_r-proximable for those r, hence f is not approximable.

Straightforward examples of approximable functions come from factorization and selection theorems. The proof of the following one is obvious.

Example 2.1. Let X be a completely regular space and $f : X \longrightarrow X$ a continuous function which factorizes continuously through a convex polyhedron P:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd}$$

Then f is approximable with respect to any underlying uniformity $\mathcal{V}(X)$ for X.

Example 2.2. Let E be a Fréchet space and $F : E \longrightarrow 2^E$ closed, compact and lower semi-continuous and suppose F has convex values. Then F is approximable.

Proof. To show the existence of a triangle (2.1) choose $(X', V) = \gamma \in \Gamma(E)$.

As a metrizable space E is paracompact and F has closed images since its graph is closed. Moreover, as a finite set, X' is closed. Hence we obtain from Theorem 1.1 a continuous selection f of F such that $X' \cap Fix(F) \subseteq Fix(f)$.

Since E is locally convex and $f(E) \subseteq F(E)$ is relatively compact there exists by Theorem 1.4 a convex polyhedron $P \subseteq E$ and a Schauder projection $s : \overline{f(E)} \longrightarrow P$ such that $(s, id)(\overline{f(E)}) \subseteq V$ and $x = s(x)$ for $x \in X' \cap Fix(F)$. We claim

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[d, "f"'] \arrow[dr, "sf"] & \arrow[d, phantom, "\gamma"] & E \\ \overline{f(E)} \arrow[r, "s"'] & P \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{2.4}$$

Preciseness on X' follows from $\iota s f|_{X' \cap Fix(F)} = id$ and since $\mathcal{G}(\iota s f) \subseteq \mathcal{G}(VF)$ we infer V-proximity. $\square$

If $F = f$ is single-valued there is no need for a selection and we obtain straightforwardly

Example 2.3. Let E be a locally convex space and $f : E \longrightarrow E$ a compact continuous function. Then f is approximable.

In Example 2.2 we are in position to approximate VF instead of VFV. To give an example where the nature of V-proximity becomes more operative we examine Kakutani's fixed point theorem. Note that in Example 2.2 F is closed and compact, hence F is upper semicontinuous. Thus the following Example is a generalization of Example 2.2 for paracompact spaces.

Example 2.4. Let E be a paracompact locally convex space and $F : E \longrightarrow 2^E$ closed and compact with convex values. Then F is approximable.

Proof. To show the existence of a triangle (2.1) choose $(X', V) = \gamma \in \Gamma(E)$.

In view of Lemma 1.1, applied to $X = Y = E$ and $U = V$, we are in position to use the same technique as for the foregoing example. We infer preciseness on X', and (VV)-proximity follows from $\mathcal{G}(\iota s f) \subseteq \mathcal{G}(VVFV)$. ☐

Example 2.5. Let X be metrizable and AE(*compact metric*). Let $f : X \longrightarrow X$ be a compact and continuous function. Then f is approximable.

Proof. C. f. [DG82, Chapter II §5 (10.8) Theorem] for the following construction.

Set $K := \overline{f(X)}$. Since K is compact metric we obtain from the Arens-Eells embedding theorem, see Theorem 1.5, a homeomorphism $h : K \longrightarrow K^\infty$ where K^∞ is a compact subset of the Hilbert cube I^∞. Define $\varphi^\infty : X \longrightarrow I^\infty$ by $\varphi^\infty := \iota h f$.

Since X is AE(*compact metric*) there exists a continuous extension $\psi^\infty : I^\infty \longrightarrow X$ of $\iota h^{-1} : K^\infty \longrightarrow X$.

Fix $(X', V) = \gamma \in \Gamma$. Since $\varphi^\infty(X') \in \mathcal{A}(I^\infty)$ there exists a Schauder projection, see Theorem 1.4, $s : I^\infty \longrightarrow P$ which extends $id : \varphi^\infty(X') \longrightarrow \varphi^\infty(X')$. Hence we infer for $\varphi := s\varphi^\infty$ and $\psi := \psi^\infty \iota$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, "\varphi^\infty"'] \arrow[dr, "\varphi"] & & X \\
I^\infty \arrow[r, "s"] & P \arrow[r, hook, "\iota"] \arrow[ur, "\psi"] \arrow[u, phantom, "\gamma"] & I^\infty \arrow[u, "\psi^\infty"']
\end{tikzcd}$$

and have shown approximability. ☐

Remark 2.2. We remark for later purpose that Example 2.5 can be generalized to the situation where $X = \cup_{i=1}^{\infty} X_i$ with $X_i \subseteq \mathring{X}_{i+1}$, $i \in \mathbb{N}$ and all X_i are AE(*compact metric*). In fact $\overline{f(X)}$, being compact, must be contained in one of the X_i and the proof of Example 2.5 goes through as above.

Since every AR(*metric*) is AE(*compact metric*) and metrizable we obtain immediately

Corollary 2.1. *Let X be AR(metric) and $f : X \longrightarrow X$ a compact and continuous function. Then f is approximable.*

Clearly Examples 2.2 to 2.5 are classical approximation results and the approximation takes place on the whole space E and not only on the fixed points of the approximating functions ψ and φ. We refer to Chapter 4 for non-classical application of our approximation concept.

By the Brouwer fixed point theorem the maps F of the above examples all have fixed points. If we suppose compactness of F this is also true for arbitrary approximable maps. More precise is

Theorem 2.1. *Let $F : X \longrightarrow 2^X$ be approximable and $\overline{F(X)}$ compact. Then F has at least one fixed point $x_0 \in F(x_0)$.*

Proof. Fix $V \in \mathcal{V}(X)$ and $(\psi, \varphi) \in (F, V)$. By the Brouwer fixed point theorem $\varphi\psi$ has a fixed point $p_V \in P$, thus $x_V := \psi(p_V)$ is a fixed point of $\psi\varphi$ which is also a fixed point of VFV by V-proximity. By compactness of $\overline{F(X)}$ the net $(x_V)_V$ has at least one cluster point x_0 which is, since $\mathcal{G}(F)$ is closed and $X \times X$ completely regular, a fixed point of F. □

The main difference between factorization techniques and our concept of proximity and preciseness is that factorization is mainly a quality of the underlying space X where our access relies more on the interrelation between the space X and the map F. We substantiate this fact by examples of maps whose approximability comes from the simple nature of their fixed point-set.

Proposition 2.1. *Let $F : X \longrightarrow 2^X$ be closed and having precisely one fixed point. Then F is approximable.*

Proof. Let x_0 be the fixed point of F. For all $\gamma \in \Gamma(X)$ we obtain the triangle

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "x_0"'] & \arrow[d, phantom, "\gamma"] & X \\ & \{x_0\} \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

where ι is the embedding. □

We emphasize that Proposition 2.1 is independent of the underlying uniformity of X.

Immediate consequences of Proposition 2.1 come from constant functions and the Banach fixed point theorem, see Theorem 1.12.

Corollary 2.2. *Every constant function is approximable.*

Corollary 2.3. *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ contractive. Then f is approximable.*

Note that approximability of F does not require the existence of a fixed point $x_0 \in F(x_0)$, see e. g. Remark 2.1.

In view of Theorem 2.1 the lack of compactness of the space $[0, \infty)$ is responsible for the unpleasant behavior of the function f considered in Remark 2.1. Indeed, the Alexandroff compactification $[0, \infty]$ of $[0, \infty)$ and the continuous extension of f would provide the fixed point ∞.

To avoid such problems one can examine only compact spaces. In view of Corollary 2.3 this is unacceptable.

2.2 Regular maps

The existence of a convergent subnet of the approximating fixed points $(x_V)_V$ in the proof of Theorem 2.1 follows from compactness of the underlying space X. We claim now the existence of such a subnet in general.

There exists a natural partial ordering on $\Gamma(X) = \mathcal{A}(X) \times \mathcal{V}(X)$ given by

$$(X_1', V_1) = \gamma_1 \geq \gamma_2 = (X_2', V_2) :\Longleftrightarrow X_1' \supseteq X_2' \text{ and } V_1 \subseteq V_2,$$

for all $\gamma_1, \gamma_2 \in \Gamma(X)$. This ordering directs $\Gamma(X)$. Observe that if $\hat{\mathcal{V}}(X)$ is another base for the given uniformity then $\mathcal{A}(X) \times \mathcal{V}(X)$ is, as a subnet, cofinal in $\mathcal{A}(X) \times \hat{\mathcal{V}}(X)$ and vice versa. Thus the following definition is independent of the choice of the base $\mathcal{V}(X)$.

Definition 2.2. We call an approximable map $F : X \longrightarrow 2^X$ *regular* if it holds the *subnet condition*, i. e. for each subnet $\Gamma' \leq \Gamma(X)$, $(\psi_{\gamma'}, \varphi_{\gamma'}) \in (F, \gamma')$ and fixed points $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})(x_{\gamma'})$ there exists a subnet $\Gamma'' \leq \Gamma'$ such that $(x_{\gamma''})_{\gamma''}$ converges to some $x \in X$.

Short hand for we will make use of the slogans *approximability* and *subnet condition* as well as those of proximity and preciseness.

The above x is a fixed point of F. Indeed, observe that each fixed point $x_{\gamma''}$ of $V_{\gamma''}FV_{\gamma''}$ is also fixed point of $\tilde{V}F\tilde{V}$ whenever $V_{\gamma''} \subseteq \tilde{V}$. Observe furthermore that Γ'' is cofinal in $\Gamma(X)$. Thus each cluster point of $(x_{\gamma''})_{\gamma''}$ belongs to $\cap_{\gamma''\in\Gamma''} Fix(V_{\gamma''}FV_{\gamma''})$ which is, by (1.3) intersected with the diagonal, equal to $Fix(F)$.

The proof of the following statements are obvious.

Proposition 2.2. *Every approximable compact map $F : X \longrightarrow 2^X$ is regular.*

Remark 2.3. We already observed, see Remark 2.1, that approximability depends on the choice of the uniformity of the underlying space X. The same is true for the subnet condition of regularity. See Remark 5.1 below for an example.

Theorem 2.2. *Let $F : X \longrightarrow 2^X$ be regular. Then F has at least one fixed point $x_0 \in F(x_0)$.*

In view of the above proposition fixed point-free approximable maps are not regular. Thus functions like $f(x) = x + \frac{x}{1+x}$, $x \in [0, \infty)$ are not under consideration anymore. We have to pay for that: Proposition 2.1 fails if one replaces 'approximable' by 'regular'. Indeed, consider $f : [0, \infty) \longrightarrow [0, \infty)$ given by $f(x) := \min\left\{\frac{3}{2}x, x + \frac{1}{1+x}\right\}$. We claim that f is approximable (with respect to the natural uniformity of $[0, \infty)$): for fixed $\gamma \in \Gamma([0, \infty))$ we infer $\left(\min\left\{\frac{3}{2}id, n\right\}, \iota\right) \in (f, \gamma)$ for sufficiently large $n \in \mathbb{N}$ since f has precisely one fixed point $x_0 = 0$. But $n = (\iota \min\{(2/3)id, n\})(n)$ for all $n \in \mathbb{N}$ and $(n)_n$ has no cluster point in $[0, \infty)$, hence f cannot be regular.

However constant and contractive functions are of interest as well as before.

Proposition 2.3. *Every constant function is regular.*

Proof. From Corollary 2.2 we know that f is approximable.

To prove the subnet condition let $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})(x_{\gamma'})$ be as in Definition 2.2 where $\gamma' = (X'_{\gamma'}, V_{\gamma'})$. Since each $x_{\gamma'}$ is a fixed point of $V_{\gamma'}fV_{\gamma'}$ and this map is identical with the (set-valued) constant map $V_{\gamma'}(x_0)$ the $x_{\gamma'}$ are forced to tend to x_0, hence they converge. □

Proposition 2.4. *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ contractive and bounded. Then f is regular.*

Proof. From Corollary 2.3 we know that f is approximable. Let K be a Lipschitz constant for f such that $K < 1$.

As in the last proof assume $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})(x_{\gamma'})$. We claim $(x_{\gamma'})_{\gamma'}$ has a subnet $(x_{\gamma''})_{\gamma''}$ which is Cauchy. This is, in view of the completeness of (X, d), sufficient for the convergence of $(x_{\gamma''})_{\gamma''}$. Fix $0 < \varepsilon \leq 1$. Since X is metric there exists $\gamma'_0 \in \Gamma'$ such that $V_{\gamma'} \subseteq V_\varepsilon$ for all $\gamma' \geq \gamma'_0$. Observe that for $\gamma' \geq \gamma'_0$ all $x_{\gamma'}$ are fixed points of $V_{\gamma'_0}fV_{\gamma'_0}$. Thus $(x_{\gamma'})_{\gamma'\geq\gamma'_0}$ is bounded since f is bounded. Since K is a Lipschitz constant for f we infer $d(x_{\gamma'_1}, x_{\gamma'_2}) \leq 2\varepsilon + K(d(x_{\gamma'_1}, x_{\gamma'_2}) + 2\varepsilon)$ for all $\gamma'_1, \gamma'_2 \geq \gamma'_0$ and, inductively for all $n \in \mathbb{N}$,

$$d(x_{\gamma'_1}, x_{\gamma'_2}) \leq 2\varepsilon(1 + K)\sum_{i=0}^{n-1} K^i + K^n d(x_{\gamma'_1}, x_{\gamma'_2}).$$

Letting $n \to \infty$ we estimate by means of the boundedness of $(x_{\gamma'})_{\gamma'\geq\gamma'_0}$

$$d(x_{\gamma'_1}, x_{\gamma'_2}) \leq 2\varepsilon\frac{1+K}{1-K}, \quad \gamma'_1, \gamma'_2 \geq \gamma'_0.$$

Since for any $\gamma' \in \Gamma'$ the above γ'_0 can be chosen such that $\gamma'_0 \geq \gamma'$ we obtain my means of $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$ a subnet $\Gamma'' = \mathbb{N} \times \Gamma'$ of Γ' such that the assigned $(x_{\gamma'_0})_{\gamma'_0}$ are Cauchy. □

In view of the above examples one may suppose at a first glance that regularity would imply upper semicontinuity. We complete this chapter with an example of a regular discontinuous function.

Example 2.6. Let $X = \mathbb{R}$ be the real line and $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then f is regular.

Proof. Define $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ by $\varphi(x) := \operatorname{sgn}(x) \min\{x^2, 1\}$, $x \in \mathbb{R}$. Approximability follows from

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "f"] \arrow[dr, "\varphi"'] & \gamma & \mathbb{R} \\ & {[-1,1]} \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

which holds for all $\gamma \in \Gamma$.

The subnet condition follows straightforwardly from the fact that for each $r > 0$ $Fix(V_r f V_r)$ is bounded. □

2.3 Regularity on closed subsets

Our main goal is to formulate Leray Schauder theory for our class of regular maps. Therefore we have to generalize our concept of proximity and preciseness to maps $F : A \longrightarrow 2^X$ defined on closed subsets A of X.

In addition to the uniform space $(X, \mathcal{V}(X))$ in what follows let A be a non-empty closed subset of X endowed with its relative uniformity which is given by the trace of $\mathcal{V}(X)$ in $A \times A$. I. e., we obtain a base of this relative uniformity by $\{V_A := V \cap (A \times A)\,;\, V \in \mathcal{V}(X)\}$.

The concept of triangle generalizes immediately to closed $A \subseteq X$. Recall that $\mathcal{A}(X)$ denotes the system of all finite subsets of X and $\Gamma(X) = \mathcal{A}(X) \times \mathcal{V}(X)$.

Consider a closed map $F : A \longrightarrow 2^X$ and fix $(X', V) = \gamma \in \Gamma(X)$. By a *triangle*

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \gamma & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd} \tag{2.5}$$

we understand the following properties of φ, ψ and P:

(i) P is a convex polyhedron and $\varphi : A \longrightarrow P$, $\psi : P \longrightarrow X$ are continuous functions,

(ii) $Fix(\psi\varphi) \subseteq Fix(VFV_A)$ (V*-proximity*),

(iii) $Fix(F) \cap X' \subseteq Fix(\psi\varphi)$ (*preciseness on* X').

Note that (iii) becomes superfluous if $X' \cap A = \emptyset$. We continue to use notation (2.3) with respect to the triangle (2.5) and associate to each $\gamma \in \Gamma(X)$ its defining $X'_\gamma \in \mathcal{A}(X)$ and $V_\gamma \in \mathcal{V}(X)$.

Definition 2.2 generalizes in an obvious way to

Definition 2.3. We call a closed map $F : A \longrightarrow 2^X$ *regular* if F is approximable, $(F, \gamma) \neq \emptyset$ for all $\gamma \in \Gamma(X)$, and holds the *subnet condition*, for any subnet $\Gamma' \leq \Gamma(X)$, $(\psi_{\gamma'}, \varphi_{\gamma'}) \in (F, \gamma')$ and fixed points $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})x_{\gamma'}$ there exists a subnet $\Gamma'' \leq \Gamma'$ such that $(x_{\gamma''})_{\gamma''}$ converges to some $x \in A$.

As before the above x turns out to be a fixed point of F.

Theorem 2.3. *Suppose that X, A or F is compact. Then F is regular iff it is approximable.*

Proof. If X or A is compact the subnet condition is obviously fulfilled. In case of compact $\overline{F(A)}$ note that $Fix(VFV_A) \subseteq (VF)(A)$ thus the $(x_{\gamma'})_{\gamma'}$ of the subnet condition tend to a compact set. Hence there exists a convergent subnet $(x_{\gamma''})_{\gamma''}$ of $(x_{\gamma'})_{\gamma'}$. $\square$

The approximability condition can turn out to be redundant, too.

Theorem 2.4. *Let $F : A \longrightarrow 2^X$ be closed and suppose that $Fix(F)$ is at most a singleton. If $A = X$ suppose in addition that $Fix(F) \neq \emptyset$. Then F is regular iff F holds the subnet condition.*

Proof. Approximability follows from the triangle

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "x_0"'] & & X \\ & \{x_0\} \arrow[ur, hook, "\iota"'] \arrow[u, phantom, "\gamma"] & \end{tikzcd}$$

where x_0 the fixed point of F or an element of $X \setminus A$ if F is fixed point-free, respectively. $\square$

Let $F : X \longrightarrow 2^X$ be a closed map having precisely one fixed point. Theorems 2.3 and 2.4 tell us that a restriction $F|_A : A \longrightarrow 2^X$ is regular provided it is compact. In particular we obtain

Example 2.7. Let (X, d) be a complete metric space and suppose a function $f : X \longrightarrow X$ holds

$$d(f(x), f(y)) < d(x, y), \quad x \neq y, \quad x, y \in X.$$

If $f|_A : A \longrightarrow X$ is compact for some non-empty closed $A \subseteq X$ then $f|_A : A \longrightarrow X$ is regular.

It is worth to mention that Propositions 2.3 and 2.4 also hold for closed subsets $A \subseteq X$, i. e. constant and contractive functions, defined on closed subsets of X, are regular.

Proposition 2.5. *Let $F : A \longrightarrow 2^X$ be a closed compact map. Suppose each $X' \in \mathcal{A}(A)$ can be connected by a simple arc in X. I. e. for each $\{x_1, \ldots, x_n\} \in \mathcal{A}(A)$ there exists, modulo a permutation of the x_i, an injective continuous function $c : [1, n] \longrightarrow X$ such that $c(i) = x_i$, $i = 1, \ldots, n$.*

Then $F : A \longrightarrow 2^X$ is regular iff either $A \neq X$ or $A = X$ and F has at least one fixed point.

Proof. By Theorem 2.3 it is sufficient to show approximability of F since F is compact and in view of Theorem 2.4 only the case $Fix(F) \neq \emptyset$ is non-trivial.

Fix $(X', V) = \gamma \in \Gamma(X)$. We can assume $\{x_1, \ldots, x_n\} = X' \subseteq Fix(F)$ and X' is non-empty. Let $c : [1, n] \longrightarrow X$ be an arc as in the hypothesis and define $\psi : [1, n] \longrightarrow X$ by

$$\psi(t) := c\left((t - \lfloor t \rfloor)^2 + \lfloor t \rfloor\right), \quad t \in [1, n],$$

where $\lfloor \cdot \rfloor$ is the floor-function.

Since $c^{-1}|_{A \cap c(I)} : A \cap c(I) \longrightarrow [1, n]$ is continuous and $[1, n]$ homeomorphic to I there exists a continuous extension $\varphi : A \longrightarrow [1, n]$ of c^{-1} by the Tietze extension theorem, see Theorem 1.2.

We claim

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\gamma"] & X \\ & {[1,n]} \arrow[ur, "\psi"'] & \end{tikzcd} \tag{2.6}$$

In fact, $[1, n]$ is a convex polyhedron and φ, ψ are continuous.

Moreover, since φ is injective on $A \cap \psi([1, n])$, x is a fixed point of $(\psi\varphi)$ iff $\varphi(x)$ is a fixed point of $(\varphi\psi)|_{\psi^{-1}(A)}$. Since $(\varphi\psi)|_{\psi^{-1}(A)}(t) = (t - \lfloor t \rfloor)^2 + \lfloor t \rfloor$ and $t = (t - \lfloor t \rfloor)^2 + \lfloor t \rfloor$ iff $t \in \{1, \ldots, n\}$ we infer $Fix(\psi\varphi) = \{x_1, \ldots, x_n\}$ from the choice of the arc, hence (2.6) follows immediately. ☐

Example 2.8. Since spheres $X = S^n$ fulfill the above hypothesis on X we infer for any $n \in \mathbb{N}$ that $F : A \longrightarrow 2^{S^n}$ is regular iff either $A \neq S^n$ or $A = S^n$ and F has at least one fixed point.

The same is true for all n-tori $\times_{i=1}^{n} S^1$.

Observe that in particular the embeddings $\iota : A \longrightarrow S^n$, $n \in \mathbb{N}$ are regular.

Indeed, every sphere S^n is arcwise connected provided that $n \in \mathbb{N}$. If $n = 1$ we can choose a segment which connects the finite set under consideration and apparently defines a simple arc. For $n > 1$ the existence of a connecting simple arc follows now by a suitable grading of the x_i and an inductive argumentation.

Remark 2.4. To emphasize that we approximate fixed point sets and not spaces and maps observe that, as a polyhedron, S^n belongs to ANR(compact metric). Hence, by the Lefschetz fixed point theorem, see Theorem 1.13, f must have a fixed point if $\Lambda(id_{S^n}) \neq 0$. But since $H_q(S^n) = \mathbb{Q}$ for $q = 0, n$ and all other homology groups are trivial we infer $\Lambda(id_{S^n}) = 0$ iff n is odd and it turns out that for odd n the identity is not determined as a function with fixed points by the Lefschetz fixed point theorem.

Remark 2.5. Proposition 2.5 is far from a representation theorem for regular maps in arcwise connected compact spaces. E. g. consider the compact space X that is given by three copies of the unit interval $I = [0, 1]$ which are glued at their zeros. By Proposition 2.9, see below, the identity $id : X \longrightarrow X$ is regular despite the fact that the hypothesis of Proposition 2.5 fail. Note that X is a fixed point space, see e. g. [EF78, Satz 2.2.7].

Moreover we obtain from Proposition 2.5 a simple proof of regularity of a compact closed map $F : A \longrightarrow 2^K$ defined on a closed convex subset K of a HTVS.

Corollary 2.4. *Let K be a closed convex subset of a HTVS E and $F : A \longrightarrow 2^K$ a compact closed map. Then the following statements are equivalent:*

(i) $F : A \longrightarrow 2^K$ is regular,

(ii) either $A \neq K$ or $A = K$ and F has at least one fixed point,

(iii) for each $\gamma \in \Gamma$ there exists a triangle

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \gamma & K \\ & I \arrow[ur, "\iota"'] & \end{tikzcd}$$

Proof. In view of (2.6) and since I is homeomorphic to $[1, n]$ is it sufficient to show the hypothesis of Proposition 2.5. Hence we have to show that each $X' \in \mathcal{A}(A)$ can be connected by a simple arc in K.

Indeed, fix $X' \in \mathcal{A}(A)$. If X' is empty or a singleton there is nothing to show. Otherwise let $K' := conv\, X'$ be the convex hull of X'. Then K' is a non-empty convex subset of K and contains a point, say x_0, which is an inner point of K' relative to the affine hull E' of K'. Let $\mu : E' \longrightarrow [0, \infty)$ be the continuous sublinear functional given by $\mu(x) := \inf\{\lambda > 0\,;\, \lambda x + (1 - \lambda)x_0 \in K'\}$, $x \in E'$. Apply Example 2.8 to those onionskins $\{x \in K'\,;\, \mu(x) = t\}$ which intersect the suitable graded X' to obtain finitely many simple arcs. Connect these arcs in a suitable way. □

In view of the above corollary there was no need so far to formulate the customized versions of our uniform approximations, e. g. of Lemma 1.1. However, in Chapter 3.3 we will need uniform approximation on certain subsets of A.

2.4 Basic properties of regularity

We first point out that whether a closed map is regular or not depends only on its behavior near the diagonal. More precise is

Lemma 2.1. *Let $F : A \longrightarrow 2^X$ and $G : A \longrightarrow 2^X$ be closed. Suppose there exists a vicinity $V_0 \in \mathcal{V}(X)$ such that*

$$Fix(VFV_A) = Fix(VGV_A), \quad V \in \mathcal{V}(X), \quad V \subseteq V_0. \tag{2.7}$$

Then

$$(F,\gamma) = (G,\gamma), \qquad \gamma \geq V_0, \qquad \gamma \in \Gamma(X) \tag{2.8}$$

and F is regular iff G is.

Proof. The statement regarding regularity follows directly from Definition 2.3 and (2.8) since the subnet condition depends only on eventually all (F,γ) and (G,γ).

It remains to prove (2.8).

V-proximity (of a pair (ψ,φ)) with respect to F apparently depends only on $Fix(VFV_A)$. Hence, by (2.7), for eventually all $V \in \mathcal{V}(X)$ V-proximity with respect to F is equivalent to V-proximity with respect to G.

Since $F : A \longrightarrow 2^X$ is closed we infer $\mathcal{G}(F) = \cap_{V\in\mathcal{V}(X),V\subseteq V_0}\mathcal{G}(VFV_A)$ from complete regularity of X. Hence $Fix(F) = \cap_{V\in\mathcal{V}(X)}Fix(VFV_A)$ from which $Fix(F) = Fix(G)$ follows. Thus preciseness with respect to F is equivalent to preciseness with respect to$G$. □

Proposition 2.6. *Let $F : A \longrightarrow 2^X$ be regular. Then*

$$\bigcap_{\gamma\in\Gamma(X)} \bigcup_{(\psi,\varphi)\in(F,\gamma)} Fix(\psi\varphi) = Fix(F) \tag{2.9}$$

and for any closed $B \subseteq A$: $B \cap Fix(F) = \emptyset$ iff there exists $V_0 \in \mathcal{V}(X)$ and $\gamma_0 \in \Gamma(X)$ such that

$$Fix(\psi_\gamma\varphi_\gamma) \cap V_0(B) = \emptyset, \qquad (\psi_\gamma,\varphi_\gamma) \in (F,\gamma), \quad \gamma \geq \gamma_0. \tag{2.10}$$

Proof. We first prove (2.9). Since $(F,\gamma_1) \subseteq (F,\gamma_2)$ for $\gamma_1 \geq \gamma_2$ inclusion $\supseteq$ follows straightforwardly from preciseness. To see inclusion $\subseteq$ let $x \in \cap_{\gamma\in\Gamma(X)} \cup_{(\psi,\varphi)\in(F,\gamma)} Fix(\psi\varphi)$, i. e. for each $\gamma \in \Gamma(X)$ there exists $(\psi,\varphi) \in (F,\gamma)$ such that $x = (\psi\varphi)(x)$. Then $(x_\gamma)_\gamma = (x)_\gamma$ fulfills the hypothesis of the subnet condition. Hence it has a subnet $(x_{\gamma'})_{\gamma'}$ which converges to a fixed point x' of F. Since X is Hausdorff it follows $x = x'$.

To prove the equivalence assume first that the right hand side fails, i. e. for all $V \in \mathcal{V}(X)$, $\gamma \in \Gamma(X)$ there exists $\gamma' \geq \gamma$ and $x_{\gamma'} \in Fix(\psi_{\gamma'}\varphi_{\gamma'}) \cap V(B)$ for some $(\psi_{\gamma'},\varphi_{\gamma'}) \in (F,\gamma')$. Thus there exists a subnet $\Gamma' := \mathcal{V}(X) \times \Gamma(X)$ of $\Gamma(X)$ such that $(x_{\gamma'})_{\gamma'}$ satisfies the hypothesis of the subnet condition. Hence $(x_{\gamma'})_{\gamma'}$ has a convergent subnet which converges to a fixed point x of F. Since B is closed x also belongs to B. This contradicts $B \cap Fix(F) = \emptyset$.

If on the other hand $B \cap Fix(F) \neq \emptyset$ then (2.10) cannot hold in view of preciseness. □

We proceed with the construction of new regular maps from old one by purely set-theoretic constructions.

The following corollary of Lemma 2.1 shows that our approximation concept for maps $F : X \longrightarrow 2^X$ is independent of exchanging ordinate and abscissa. Its proof follows directly from Lemma 2.1. The succeeding example follows straightforwardly from Examples 2.3 and 2.7.

Corollary 2.5. *Let $F : X \longrightarrow 2^X$ be surjective. Then F^{-1} is regular iff so is F.*

Example 2.9. Let (X, d) be a complete metric space with bounded metric d. Let $f : X \longrightarrow X$ be surjective and suppose in addition that either

(i) f is *expansive*, i. e. there exists $1 < K < \infty$ such that

$$d(f(x), f(y)) \geq K d(x, y), \quad x, y \in X \quad \text{or}$$

(ii) X is compact and

$$d(f(x), f(y)) > d(x, y), \quad x \neq y, \quad x, y \in X.$$

Then f is regular.

It is worth to mention that we cannot make statements about regularity of F^{-1} if a general surjective regular map $F : A \longrightarrow 2^X$ is under consideration. We loose control about the fixed point set of the approximating functions $(\psi, \varphi) \in (F, \gamma)$ since we are forced to extend φ whenever the domain of definition increases.

The next results are negative in character.

Proposition 2.7. *Let $F : A \longrightarrow 2^X$ and $G : A \longrightarrow 2^X$ be regular. Then $F \cup G$ and $\overline{X \setminus F}$ are in general not regular. Furthermore, even if $F \cap G$ is non-empty-valued and FG defined and closed these maps need not be regular.*

Proof. Concerning union and complement consider finite spaces X with discrete uniformity and $A = X$. Constant functions F and G are regular by Proposition 2.3. If X contains more than two (three) points than the fixed point set of $F \cup G$ $(\overline{X \setminus F})$ contains more than one element. Hence $F \cup G$ $(\overline{X \setminus G})$ cannot be regular by Theorem 2.6, see Chapter 2.6 below.

For counterexamples to intersection and concatenation we consider the unit interval I. First let F be given by $F(x) = 1$, $0 \leq x < \frac{1}{2}$, $F(\frac{1}{2}) = I$, $F(x) = 0$, $\frac{1}{2} < x \leq 1$ and G be the constant (set-valued) map $G = \{0, 1\}$. Since I is compact by Theorem 2.3 F and G are regular iff they satisfy the approximability condition. Approximability of F follows from Example 2.4. Regarding G take a look at

$$\begin{tikzcd} I \arrow[rr, "\{0,1\}"] \arrow[dr, "x \mapsto x^2"'] & {} \arrow[d, phantom, "\gamma"] & I \\ & I \arrow[ur, "id"'] & \end{tikzcd}$$

which holds for all $\gamma \in \Gamma(I)$. Thus F and G are regular. Since their intersection $F \cap G$ is fixed point-free it cannot be regular in view of Theorem 2.2.

Now let $F : I \longrightarrow 2^I$ be given by $F(x) = \frac{1}{2}$, $0 \le x < \frac{1}{2}$, $F\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, $F(x) = 0$, $\frac{1}{2} < x \le 1$ and $G = x \mapsto \sqrt{x}$. Again F and G are regular iff they satisfy the approximability condition. Thus F is regular by Proposition 2.1 and G by Example 2.1. As above, the fixed point-free FG cannot be regular. □

We are now concerned with the question whether regularity of a map $F : A \longrightarrow 2^X$ is preserved if it is possible to embed its graph into a subspace of $A \times X$. In general regularity fails to be preserved. E. g. consider, as in the proof of Proposition 2.7, the constant map $G = \{0, 1\}$ on the unit interval $A := X := I$. Remove some not fixed points, e. g. an open interval $I' \subseteq (0, 1)$, from I. Then there exists no longer an arc from 0 to 1 in $I \setminus I'$. Hence, in view of Theorem 2.6, see below, the restricted F cannot be regular.

Proposition 2.8. *Let X be a compact subspace of a uniform space Y and $F : A \longrightarrow 2^X$ regular. Then $\iota F : A \longrightarrow 2^Y$ is regular.*

Proof. Since X is closed in Y and $\mathcal{G}(F)$ is closed in $A \times X$ so is $\mathcal{G}(F)$ closed in $A \times Y$.

By Theorem 2.3 and compactness of F we have to show approximability of ιF.

Let $\gamma_Y = (X', V_Y) \in \Gamma(Y)$. Since $V_X := V_Y \cap (X \times X)$ is a vicinity in $\mathcal{V}(X)$ we infer $\gamma_X := (X' \cap X, V_X) \in \Gamma(X)$. Hence $(\iota\psi, \varphi) \in (\iota F, \gamma_Y)$ provided that $(\psi, \varphi) \in (F, \gamma_X)$. Since F is approximable (F, γ_X) is non-empty and so $(\iota F, \gamma_Y)$ is non-empty, too. Hence ιF is approximable. □

2.5 Restriction and extension of regular maps

The next corollary treats restrictions of regular maps. First we need a preliminary topological statement.

A *closed pair* (B, A) in a topological space X is a pair of subsets $A \subseteq B \subseteq X$ such that A and B are closed. A closed pair (B, A) is said to be a *compact pair* if B is compact.

Lemma 2.2. *Let (B, A) a closed pair in a uniform space $(X, \mathcal{V}(X))$ such that A is compact. Then for any vicinity $V \in \mathcal{V}(X)$ there exists a vicinity $U \in \mathcal{V}(X)$ such that for all $x \in A$, $y \in B \setminus A$*

$$\Big((x, y) \in U \quad \Rightarrow \quad \exists\, z \in \partial_B A \,:\, (x, z) \in V \text{ and } (y, z) \in V\Big). \tag{2.11}$$

Proof. If A or $B \setminus A$ is empty then there is nothing to show. If $\partial_B A$ is empty there exists $U \in \mathcal{V}(X)$ that separates A and $B \setminus A$ and the premise in (2.11) always fails.

Assume that $\partial_B A$ is non-empty. We prove indirectly. We can assume the existence of two nets $(x_U)_{U \in \mathcal{V}(X)} \subseteq A$ and $(y_U)_{U \in \mathcal{V}(X)} \subseteq B \setminus A$ such that $(x_U, y_U) \in U$, $U \in \mathcal{V}(X)$ and $(x_U)_{U \in \mathcal{V}(X)}$ converges to $x \in A$. Furthermore we can assume that there exists $V_0 \in \mathcal{V}(X)$ such that, without loss of generality, $y_U \notin V_0(\partial_B A)$, $U \in \mathcal{V}(X)$. Since $(y_U)_{U\in\mathcal{V}(X)}$ converges to x, too, we infer $x \in \partial_B A$ which is a contradiction. □

Lemma 2.3. *Let (B, A) be a closed pair in a uniform space X such that A or $B \setminus A$ is compact. Let $F : B \longrightarrow 2^X$ be upper semicontinuous on $\partial_B A$. Then for all $\gamma_A \in \Gamma(X)$ there exists $\gamma_B \in \Gamma(X)$ such that*

$$\Big((\psi, \varphi) \in (F, \gamma_B) \quad \Rightarrow \quad (\psi, \varphi|_A) \in (F|_A, \gamma_A)\Big). \tag{2.12}$$

Proof. Fix $(X', V) = \gamma_A \in \Gamma(X)$. Choose $V_1 \in \mathcal{V}(X)$ such that $V_1V_1 \subseteq V$. Since F is uniformly upper semicontinuous on the (compact) set $\partial_B A$ there exists $V_2 \in \mathcal{V}(X)$ such that $F((V_2)_B(z)) \subseteq V_1F(z)$ for all $z \in \partial_B A$. Apply Lemma 2.2 to obtain $V_3 \in \mathcal{V}(X)$ such that

$$\Big((x, y) \in V_3 \quad \Rightarrow \quad \exists\, z \in \partial_B A \,:\, (x, z) \in V_2 \text{ and } (y, z) \in V_2\Big),$$

define $U := V_1 \cap V_3$ and $\gamma_B := (X', U)$. To show (2.12) choose $(\psi, \varphi) \in (F, \gamma_B)$. Note first that preciseness on X' for $(\psi, \varphi|_A)$ follows directly from preciseness of (ψ, φ). It remains to show V-proximity for $(\psi, \varphi|_A)$. By (2.2) this follows from $\mathcal{G}(\psi\varphi|_A) \cap \Delta(A) \subseteq \mathcal{G}(UFU_B) \cap (A \times X)$ and

$$\begin{aligned}
&\mathcal{G}(UFU_B) \cap (A \times X)\\
&= \mathcal{G}(UF|_AU_A) \,\cup\, \{(x, x')\,;\, \exists\, y \in B \setminus A \,:\, x \in A,\, (y, x) \in U,\, x' \in (UF)(y)\}\\
&\subseteq \mathcal{G}(VF|_AV_A) \,\cup\, \{(x, x')\,;\, \exists\, y \in B \setminus A \,:\, x \in A,\, (y, x) \in V_3,\, x' \in (V_1F)(y)\}\\
&\subseteq \mathcal{G}(VF|_AV_A) \,\cup\, \{(x, x')\,;\, \exists\, z \in \partial_B A \,:\, x' \in (V_1F)(V_{2_B}(z))\}\\
&\subseteq \mathcal{G}(VF|_AV_A) \,\cup\, \{(x, x')\,;\, \exists\, z \in \partial_B A \,:\, x' \in (VF)(z)\}\\
&= \mathcal{G}(VF|_AV_A).
\end{aligned}$$

□

In view of Theorem 2.3 and Lemma 2.3 the proof of the following corollary is now obvious.

Corollary 2.6. *Let (B, A) be a closed pair in a uniform space $(X, \mathcal{V}(X))$ such that A is non empty and compact. Let $F : B \longrightarrow 2^X$ be regular. Then $F|_A : A \longrightarrow 2^X$ is regular, too.*

What follows is a fundamental extension theorem which power comes into operation in Chapter 3.3.

Theorem 2.5. *Let X be a contractible and locally contractible metrizable space. Let $F : A \longrightarrow 2^X$ be regular and suppose that $F|_{\partial A} = x_0$ for an inner point x_0 of A. Suppose that F is uniformly upper semicontinuous on ∂A. Finally suppose*

$$\forall\, \gamma \in \Gamma \quad \exists\, (\psi, \varphi) \in (F, \gamma) : \quad \mathcal{G}(\psi\varphi|_{\partial A}) \subseteq \mathcal{G}(V_\gamma F(V_\gamma)_A)) \tag{2.13}$$

where V_γ is associated to γ.

Then $\hat{F} : X \longrightarrow 2^X$, given by

$$\hat{F}(x) := \begin{cases} F(x), & x \in A \\ x_0, & x \in X \setminus A, \end{cases} \tag{2.14}$$

is regular.

Proof. If $A = X$ there is nothing to show, thus we assume that $X \setminus A$ is non-empty. $\hat{F}$ is apparently closed and we have to show approximability and the subnet condition for $\hat{F}$.

Since $F|_{\partial A} = x_0 \in \mathring{A}$ and F is uniformly upper semicontinuous on ∂A there exists an open $U \in \mathcal{V}(X)$ and $V_0 \in \mathcal{V}(X)$ such that

$$U(\partial A) \cap (V_0(x_0) \cup Fix(F)) = \emptyset. \tag{2.15}$$

To prove approximability fix $(X', V) = \gamma \in \Gamma(X)$. Without loss of generality assume $V \subseteq V_0$. We have to show the existence of a triangle

$$\begin{tikzcd} X \arrow[rr, "\hat{F}"] \arrow[dr, "\hat{\varphi}"'] & \gamma & X \\ & \hat{P} \arrow[ur, "\hat{\psi}"'] & \end{tikzcd}$$

and start with the construction of the function $\hat{\varphi}$.

By Lemma 1.2 there exists $V' \in \mathcal{V}(X)$, assumed to be open and contained in V, such that the assertion of the Lemma holds with $O = V(x_0)$ and $Q = V'(x_0)$.

Since $F = x_0$ on ∂A and F is uniformly upper semicontinuous on ∂A we infer from (2.13) a triangle

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \gamma' & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd}$$

such that

$$\gamma' \geq \gamma, \quad \gamma' \geq U \quad \text{and} \quad (\psi\varphi)(\partial A) \subseteq V'(x_0). \tag{2.16}$$

Set

$$\tilde{P} := \psi^{-1}(V'(x_0)) \tag{2.17}$$

and define $\psi' : \tilde{P} \uplus \{*\} \longrightarrow V'(x_0)$ by $\psi' = \psi$ on $\tilde{P}$ and $\psi'(*) = x_0$. Identify now $*$ with the tip of *cone* $\tilde{P}$. From Lemma 1.2 we obtain an extension $\psi'' :$ *cone* $\tilde{P} \longrightarrow V(x_0)$ of ψ' which itself can be extended by Lemma 1.4 to $\hat{\psi} :$ *cone* $P \longrightarrow X$ such that $\hat{\psi}$ is also an extension of ψ. Set $\hat{P} :=$ *cone* P.

We switch now to the construction of $\hat{\varphi}$. Since U and $V'(x_0)$ are open by (2.17), (2.16)

$$\tilde{A} := \varphi^{-1}(\tilde{P}) \cap U_A(\partial A) \tag{2.18}$$

is an open neighborhood, relative A, of ∂A.

Since X is normal there exists an Urysohn function $\lambda : X \longrightarrow I$ such that $\lambda = 1$ on $X \setminus \mathring{A}$ and $\lambda = 0$ on $A \setminus \tilde{A}$. Define $\hat{\varphi} : X \longrightarrow \hat{P}$ by $\hat{\varphi}(x) := (1 - \lambda(x))\varphi(x) + \lambda(x)*$, $x \in X$.

We claim $(\hat{\psi}, \hat{\varphi}) \in (F, \gamma)$.

Apparently $\hat{\psi}$ and $\hat{\varphi}$ are continuous and $\hat{P}$ is a convex polyhedron. To calculate $Fix(\hat{\psi}\hat{\varphi})$ we analyze the graph of $\hat{\psi}\hat{\varphi}$. Consider the partition

$$X = (X \setminus A) \cup \tilde{A} \cup (A \setminus \tilde{A}).$$

If $x \in X \setminus A$ then $\lambda(x) = 1$, thus $(\hat{\psi}\hat{\varphi})(x) = \hat{\psi}(*) = \psi'(*) = x_0$. If $x \in A \setminus \tilde{A}$ then $\lambda(x) = 0$, hence $\hat{\varphi}(x) = \varphi(x) \in P$ and $(\hat{\psi}\hat{\varphi})(x) = \hat{\psi}(\varphi(x)) = \psi(\varphi(x))$. Thus

$$\hat{\psi}\hat{\varphi} = \psi\varphi \quad \text{on} \quad A \setminus \tilde{A}. \tag{2.19}$$

If $x \in \tilde{A}$ then $\hat{\varphi}(x) \in \mathit{cone}\,\tilde{P}$, thus $(\hat{\psi}\hat{\varphi})(x) = \psi''(\mathit{cone}\,\tilde{P}) \in V(x_0)$. Hence, since $V \subseteq V_0$ and by (2.15),

$$Fix(\hat{\psi}\hat{\varphi}) \subseteq A \setminus \tilde{A}. \tag{2.20}$$

We claim V-proximity of $(\hat{\psi}, \hat{\varphi})$ with respect to $\hat{F}$. Indeed, by (2.19) and (2.20),

$$Fix(\hat{\psi}\hat{\varphi}) = Fix(\hat{\psi}\hat{\varphi}) \cap (A \setminus \tilde{A}) \subseteq Fix(\psi\varphi)$$

and from V-proximity of (ψ, φ) with respect to F

$$Fix(\psi\varphi) \subseteq Fix(VFV_A) \subseteq Fix(V\hat{F}V).$$

Since by (2.18), (2.15) $Fix(\tilde{F}) = Fix(F|_{A\setminus\tilde{A}})$ preciseness of $(\hat{\psi}, \hat{\varphi})$ on X' with respect to $\hat{F}$ follows from (2.19) and preciseness of (ψ, φ) on X' with respect to F.

We switch now to the subnet condition. Assume $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})(x_{\gamma'})$ for some $(\psi_{\gamma'}, \varphi_{\gamma'}) \in (\hat{F}, \gamma')$ and $\gamma' \in \Gamma'$ with $\Gamma' \leq \Gamma$. Similar argumentation as for the proof of (2.15) shows the existence of $V_1 \in \mathcal{V}(X)$ such that $(V_1\hat{F}V_1)|_{X\setminus(A\setminus V_1(\partial A))}$ is fixed point-free. Thus $x_{\gamma'} \in A$ for $\gamma' \geq V_1$ by V_1-proximity and $(\psi_{\gamma'}, \varphi_{\gamma'}|_A)$ fulfills the approximability condition with respect to F, i. e. $(\psi_{\gamma'}, \varphi_{\gamma'}|_A) \in (F, \gamma')$ for all $\gamma' \geq V_1$. Since the regular F holds its subnet condition $(x_{\gamma'})_{\gamma'}$ must have a convergent subnet.

We have demonstrated regularity of $\hat{F}$. □

Remark 2.6. Since Theorem 2.5 is essential for the results in Chapter 3.3 we take a closer look at its topological assumptions.

Whether metrizability is a necessary assumption for Theorem 2.5 or normality is sufficient for the proof is a difficult question. We need metrizability to apply Lemma 1.4. Normality turns out to be sufficient for the proof of Theorem 2.5 if X is ANE(polyhedra). As a matter of fact Lemma 1.4 tells us that locally contractible metrizable spaces even are ANE(finite dimensional metric).

Global contractibility can be replaced by X is AE(polyhedra).

Consider now the discrete two point-space $X = \{a, b\}$. X is metrizable, compact and locally contractible but not contractible. a is an inner point of $A := \{a\}$ and $\partial A = \emptyset$. The constant fixed point-free function $b : \{a\} \longrightarrow \{a, b\}$ is regular. Following Theorem 2.5 we have to extend $F = b$ by $\hat{F}(b) := a$. But the fixed point-free $\hat{F}$ cannot be regular. As a matter of fact the proof of Theorem 2.5 shows us in a complicated way the simple fact that the discrete two point-space does not belong to AE(polyhedra).

What follows points out that local contractibility together with (2.13) is a necessary condition for Theorem 2.5.

Consider S. Kinoshitas [Kin53] example of a contractible compact space $Y \subseteq \mathbb{R}^3$ without fixed point property. I. e. there exists a fixed point-free function $g : Y \longrightarrow Y$. The space $Y = Y_1 \cup Y_2 \cup Y_3$ reads in cylinder-coordinates $\big((r, \varphi, z) \in [0, \infty) \times [0, \infty) \times \mathbb{R}$ with identifications $(r, \varphi, z) \sim (r, \varphi', z)$ iff $\varphi = \varphi' \mod 2\pi$ and $(0, \varphi, z) \sim (0, \varphi', z)\big)$ as

$$Y_1 = \left\{(r, \varphi, z)\,;\, 0 \le r < \frac{\pi}{2}, 0 \le \varphi < 2\pi, z = 0\right\}$$
$$Y_2 = \{(r, \varphi, z)\,;\, r = \arctan\varphi, 0 \le \varphi < \infty, 0 < z \le 1\}$$
$$Y_3 = \left\{(r, \varphi, z)\,;\, r = \frac{\pi}{2}, 0 \le \varphi < 2\pi, 0 \le z \le 1\right\}.$$

There will be no need to give the precise definition of g, the fixed point-free function. We refer to S. Kinoshita [Kin53] or [EF78, Beispiel 3.2.13] for the complete definition of g.

Y is metrizable, compact and contractible but not locally contractible. Choose $y_0 \in Y$ and let $h^t : Y \longrightarrow Y$ be a homotopy such that $h^0 = id$ and $h^1 = y_0$. Define $X := \mathit{cone}\, Y$, $\hat{f} : X \longrightarrow X$ by

$$\hat{f}(t, x) := \left(0, gh^{\min\{2t,1\}}(x)\right), \quad (t, x) \in I \times Y$$

and consider $A := [0, \frac{1}{2}] \times Y$ as a subset of X. We claim $f = \hat{f}|_A$ fulfills the assumptions of Theorem 2.5 except local contractibility of X and condition (2.13). Indeed, $X = \mathit{cone}\, Y$ is compact, metrizable and contractible. $x_0 := (0, g(y_0))$ is an inner point of $A = [0, \frac{1}{2}] \times Y$ in X, the boundary of A is $\partial A = \{\frac{1}{2}\} \times Y$ and $f|_{\partial A} = \hat{f}|_{\{\frac{1}{2}\} \times Y} = (0, gh^1) = (0, g(y_0)) = x_0$. f is uniformly continuous on the compact set A. Moreover f is fixed point-free. Indeed, $(t, x) = f(t, x)$ iff $(t, x) = (0, gh^{\min\{2t,1\}}(x))$ iff $t = 0$ and $x = g(x)$. But g is fixed point-free. Regularity of f follows now from Theorems 2.3 and 2.4.

The fixed point-free $\hat{f}$, which conincides with $\hat{F}$ of (2.14), cannot be regular, i. e. Theorem 2.5 fails.

In connection with Theorem 2.4 Theorem 2.5 provides an immediate

Corollary 2.7. *Let X be a contractible and locally contractible compact metrizable space. Let $F : A \longrightarrow 2^X$ be closed and suppose that $F|_{\partial A} = x_0$ where x_0 is an inner point of A. Finally suppose* (2.13). *Then F has a fixed point.*

Proof. By (2.13) the closed map F is approximable and regularity of F follows from Theorem 2.3 since X is compact. Since F is closed and maps between compact spaces F is uniformly upper semicontinuous on ∂A. Together with (2.13) the suppositions of Theorem 2.5 are given and $\hat{F}$ in (2.14) is regular. By Theorem 2.4 $\hat{F}$ has a fixed point $x_1 \in X$. Since $\hat{F} = x_0$ on $X \setminus A$ and $x_0 \in \mathring{A}$ we infer $x_1 \in X \setminus A$. Hence x_1 is also a fixed point of F. $\square$

2.6 Regularity of embeddings

If $id : X \longrightarrow X$ is regular we obtain information on the underlying Tychonoff space X. We start with a statement frequently used so far.

Theorem 2.6. *The set of fixed points $Fix(F)$ of a regular map $F : A \longrightarrow 2^X$ is compact and contained in one arc-component of X.*

Proof. To prove compactness of $Fix(F)$ we show that each net in $Fix(F)$ has a cluster point in $Fix(F)$.

Let $(x_\lambda)_\lambda$ be a net in $Fix(F)$ indexed over Λ. By preciseness for any $(\lambda, \gamma) \in \Lambda \times \Gamma(X)$ there exist $\gamma' \in \Gamma(X)$ and $(\psi_{\gamma'}, \varphi_{\gamma'}) \in (F, \gamma')$ such that $x_\lambda = (\psi_{\gamma'}\varphi_{\gamma'})(x_\lambda)$. We infer a subnet $\Gamma' := \Lambda \times \Gamma(X)$ of $\Gamma(X)$ with $x_{\gamma'} = (\psi_{\gamma'}\varphi_{\gamma'})(x_{\gamma'})$ and are in position to apply the subnet condition. Hence there exists $\Gamma'' \leq \Gamma'$ such that $(x_{\gamma''})_{\gamma''}$ converges to a fixed point of F. Since $\Gamma'' \leq \Gamma' = \Lambda \times \Gamma(X) \leq \Lambda$ this point is a cluster point of $(x_\lambda)_\lambda$.

To see that $Fix(F)$ is contained in one arc-component of X let $\{x, y\} \subseteq Fix(F)$. From preciseness we obtain a triangle

$$\begin{tikzcd} A \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \{x,y\} & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd}$$

with a convex polyhedron P. The preciseness of the above triangle reads as $x = \psi(\varphi(x))$ and $y = \psi(\varphi(y))$. Thus $t \mapsto \psi((1-t)\varphi(x) + t\varphi y)$ is defined and provides an x and y connecting arc in X. $\square$

Corollary 2.8. *Let X be a Tychonoff space and $id : X \longrightarrow X$ be regular. Then X is compact and arcwise connected.*

Proposition 2.9. *Let X be AE(polyhedra) or contractible and $A \subseteq X$ be non-empty and compact. Then the embedding $\iota : A \hookrightarrow X$ is regular.*

Proof. Since A is compact it is sufficient to show approximability of ι. Since $Fix(V\iota V_A) = Fix(\iota) = A$ proximity is trivial.

To prove preciseness choose $X' \in \mathcal{A}(X)$. We have to show that (F, X') is non-empty and can assume that X' is non-empty. If $A \cap X' = \emptyset$ then $A \neq X$ and for any $x_0 \in X \setminus A$ we infer $(\iota, x_0) \in (F, X')$ where ι is the embedding of $\{x_0\}$ into X. Otherwise consider $A \cap X'$ as vertices of a simplex Δ_n which itself embeds into a cube I^n. Denote the embedding of $A \cap X'$ by $\iota_n : A \cap X' \hookrightarrow I^n$.

By the Tietze extension theorem, see Theorem 1.2, ι_n admits a continuous extension $\varphi : A \longrightarrow I^n$. Since X is AE(polyhedra) (or in view of Lemma 1.3) the embedding $\iota_n' : A \cap X' \hookrightarrow X$ admits an extension $\psi : I^n \longrightarrow X$. Apparently $(\psi, \varphi) \in (F, X')$. □

We emphasize the differences between our concept of regularity and the classical approximation concepts like Schauder projection or admissibility (see Chapter 4.2 for the definition of the latter one).

Consider again S. Kinoshitas [Kin53] example of a compact contractible $X \subseteq \mathbb{R}^3$ without fixed point property. Proposition 2.9 tells us that $id : X \longrightarrow X$ is regular despite the fact that there exists a fixed point-free $f : X \longrightarrow X$. In other words regularity of the identity in general does not allow any statements about regularity of other functions.

By Lemma 1.4 contractibility implies AE(polyhedra) for locally contractible metric spaces. At a first glance one may suppose equivalence in Proposition 2.9 for locally contractible compact metric spaces X. I. e. regularity of all embeddings characterizes AE(polyhedra). Since the spheres S^n are not AE(polyhedra), see e. g. [Bor67, (17.6), Chapter I], this fails in view of Example 2.8.

Examples 2.1, 2.2 and 2.4 provide a variety of regular maps $F : X \longrightarrow 2^X$ for nice spaces X, e. g. convex polyhedra. If all closed maps F are assumed to be regular X turns out to be quite simple. A characterization of the one-point space in terms of regular maps is

Corollary 2.9. *A Tychonoff space X is the one-point space iff all closed $F : X \longrightarrow 2^X$ are regular.*

Proof. Sufficiency is obvious. To prove necessity assume that X contains two different points x and y. Choose an open vicinity $V \in \mathcal{V}(X)$ such that $(x, y) \notin \overline{V}$ and consider the map $F : X \longrightarrow 2^X$ given by $F|_{V(x)} = y$, $F|_{X \setminus \overline{V}(y)} = x$, and $F|_{\partial V(x)} = \{x, y\}$. F is closed and fixed point-free, hence it cannot be regular. □

3 Homotopies

We generalize the definition of regularity to homotopies. Regular homotopies can be considered as closed deformations of regular maps.

As before, in what follows $(X, \mathcal{V}(X))$ is a uniform space and $A \subseteq X$ non-empty and closed.

3.1 Regular homotopies

A *homotopy* F^t is a family of maps $F^t : A \longrightarrow 2^X$ indexed over $t \in I$.

We consider *closed homotopies* F^t, i. e. we associate to F^t a map $\tilde{F} : I \times A \longrightarrow 2^X$ by $\tilde{F}(t, x) := F^t(x)$, $x \in A$, $t \in I$ and postulate this map to be closed. Observe that for a closed homotopy F^t and fixed $t_0 \in I$ the map $F^{t_0} : A \longrightarrow 2^X$ is closed, too.

In particular any *continuous homotopy* $f^t : A \longrightarrow X$, i. e. $\tilde{f} : I \times A \longrightarrow X$ is a continuous function, is a closed homotopy.

Definition 3.1. Let $F^t : A \longrightarrow X$ be a closed homotopy and $(X', V) = \gamma \in \Gamma = \mathcal{A}(A) \times \mathcal{V}(X)$. By a *triangle*

$$\begin{tikzcd} A \arrow[rr, "F^t"] \arrow[dr, "\varphi^t"'] & \arrow[d, phantom, "\gamma"] & X \\ & P \arrow[ur, "\psi^t"'] & \end{tikzcd} \tag{3.1}$$

we understand the following properties of φ^t, ψ^t and P:

(i^t) P is a convex polyhedron and $\varphi^t : A \longrightarrow P$, $\psi^t : P \longrightarrow X$ are continuous homotopies,

(ii^t) $Fix(\psi^t \varphi^t) \subseteq Fix(V F^t V_A)$, $\quad t \in I \qquad$ (*V-proximity*),

(iii^t) $Fix(F^t) \cap X' \subseteq Fix(\psi^t \varphi^t)$, $\quad t \in I \qquad$ (*preciseness on X'*).

Again short hand for define

$$(F^t, \gamma) := \{(\psi^t, \varphi^t)\,;\, \psi^t \text{ and } \varphi^t \text{ constitute a triangle (3.1)}\}$$

and associate to each pair $(\psi^t, \varphi^t) \in (F^t, \gamma)$ its convex polyhedron P and the associated continuous functions $\tilde{\varphi} : I \times A \longrightarrow P$ and $\tilde{\psi} : I \times P \longrightarrow X$.

Definition 3.2. We call a closed homotopy $F^t : A \longrightarrow 2^X$ *regular* if F^t is *approximable*, $(F^t, \gamma) \neq \emptyset$ for all $\gamma \in \Gamma(X)$, and holds the *subnet condition*, for any subnet $\Gamma' \leq \Gamma(X)$, $(\psi^t_{\gamma'}, \varphi^t_{\gamma'}) \in (F^t, \gamma')$ and fixed points $x_{\gamma'} = \left(\psi^{t_{\gamma'}}_{\gamma'} \varphi^{t_{\gamma'}}_{\gamma'}\right)(x_{\gamma'})$, where $t_{\gamma'} \in I$, there exists a subnet $\Gamma'' \leq \Gamma'$ such that $(x_{\gamma''})_{\gamma''}$ converges to some $x \in A$.

Consider the above net Γ''. Since $I = [0,1]$ is compact there exists a subnet $\Gamma''' \leq \Gamma''$ such that $t_{\gamma'''} \longrightarrow t$ in I and $x_{\gamma'''} \longrightarrow x$. Thus, by the same argumentation as for regular maps, we infer from (1.3) applied to $\tilde{F}$ that the above x is a fixed point of the homotopy F^t, i. e. x belongs to

$$Fix((F^t)) := \{x \in A\,;\, x \in F^t(x) \text{ for some } t \in I\}. \tag{3.2}$$

Note that

$$Fix((F^t)) = \Pi_X Fix(I \times \tilde{F}) \tag{3.3}$$

where Π_X is the projection from $I \times X$ onto X.

In parallel to Theorem 2.3 the subnet condition becomes superfluous if X, A or F^t is compact where a homotopy $F^t : A \longrightarrow 2^X$ is said to be *compact* if $\tilde{F}$ is compact.

Moreover observe that any approximable homotopy F^t defines a family F^{t_0}, $t_0 \in I$ of approximable maps by fixing $t = t_0$. Hence if $A = X$ and F^t is compact each F^{t_0} is regular and $Fix(F^{t_0})$ is non-empty. In particular $Fix((F^t)) = \cup_{t_0 \in I} Fix(F^{t_0})$ is non-empty.

Proposition 2.8 and Corollary 2.6 hold for homotopies, too, and they are proved along the lines of their proofs for maps. Moreover Theorem 2.6 generalizes to homotopies: $Fix((F^t))$ is compact and arcwise connected in X. E. g. if $x \in F^{t_x}(x)$ and $y \in F^{t_y}(y)$ are fixed points of the homotopy $F^t : A \longrightarrow 2^X$ then for any $(\psi^t, \varphi^t) \in (F^t, \{x, y\})$

$$t \mapsto \psi^{(1-t)t_x + t t_y}\left((1-t)\varphi^{t_x}(x) + t\varphi^{t_y}(y)\right)$$

defines by preciseness an x and y connecting arc in X. We omit the (straightforward) proofs of the above statements.

Proposition 2.3 generalizes to

Example 3.1. Every arc $c : I \longrightarrow X$ defines a regular homotopy $c^t : A \longrightarrow X$ by

$$c^t(x) := c(t), \quad x \in A, \quad t \in I.$$

Proof. Since c is continuous c^t is closed and since $c(I)$ is compact so is c^t. Hence it is sufficient to show approximability of c^t.

Fix $\gamma \in \Gamma(X)$. We claim

$$\begin{tikzcd}
A \arrow[rr, "c^t"] \arrow[dr, "t"'] & \gamma & X \\
& I \arrow[ur, "c"'] &
\end{tikzcd}$$

where t denotes the constant function $x \mapsto t$, $x \in A$. Indeed, I is a convex polyhedron, t and c are continuous homotopies and for any $t \in I$ we infer $Fix(ct) = Fix(c(t)) = \{x \in A\,;\, x = c(t)\} = Fix(c^t)$. Thus $(c, t) \in (c^t, \gamma)$ for all $\gamma \in \Gamma$ and the homotopy c^t is approximable. $\square$

The following examples generalize Examples 2.2, 2.4, 2.5, Corollary 2.1 and Proposition 2.4 to homotopies.

Example 3.2. Let E be a Fréchet space and $F^t : A \longrightarrow 2^E$ closed and compact with convex values.

Then $F^t : A \longrightarrow 2^E$ is regular.

Proof. The subnet condition is redundant since F^t is compact and we have to show approximability of F^t.

Fix $(X', V) = \gamma \in \Gamma(X)$. Define

$$J(x') := \{t \in I\,;\, x' \in F^t(x')\}, \quad x' \in X'.$$

Since F^t is closed all $J(x')$ are compact. $\hat{X} = \cup_{x' \in X'}(J(x') \times \{x'\})$ is a pairwise disjoint union of finitely many compacta in $I \times A$ and the associated map $\tilde{F} : \hat{X} \longrightarrow 2^E$ admits the selections $x' : J(x') \times \{x'\} \longrightarrow E$, $x' \in X'$. Thus we are in position to apply Lemma 1.1 with respect to the map $\tilde{F} : I \times A \longrightarrow 2^X$. Proceeding as in the proof of Example 2.4 we infer

$$\begin{tikzcd}
A \arrow[rr, "F^t"] \arrow[d, "f^t"'] \arrow[dr, "sf^t"] & \gamma & E \\
\overline{\tilde{F}(A)} \arrow[r, "s"] & P \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

which provides approximability. $\square$

There is no need to elaborate a proof of

Example 3.3. Let X be metric and AE(*compact metric*). (E. g. X is AR (*metric*).) Then any compact continuous homotopy $f^t : A \longrightarrow X$ is regular.

To generalize Proposition 2.4 to homotopies we first define an appropriate notion of contractive families which is taken from A. Granas [Gra94]. We use a slightly generalized definition.

Definition 3.3. Let (X,d) be a complete metric space and $A \subseteq X$ non-empty and closed. A homotopy $f^t : A \longrightarrow X$ is said to be an *α-contractive family* if there exists $0 \leq \alpha < 1$ and a continuous semi-metric d' on I such that

$$d\left(f^t(x_1), f^t(x_2)\right) \leq \alpha d(x_1, x_2), \quad x_1, x_2 \in A, \quad t \in I \tag{3.4}$$

and

$$d\left(f^{t_1}(x), f^{t_2}(x)\right) \leq d'(t_1, t_2), \quad x \in A, \quad t_1, t_2 \in I. \tag{3.5}$$

Example 3.4. Every arc $c : I \longrightarrow X$ in a complete metric space (X,d) defines a 0-contractive family by Example 3.1. Indeed, (3.4) is apparently fulfilled for the induced homotopy $c^t : A \longrightarrow X$ and (3.5) follows by means of the continuous semi-metric d' given by $d'(t_1, t_2) := d(c(t_1), c(t_2))$, $t_1, t_2 \in I$.

Remark 3.1. In fact, the arcs are the main reason for our generalization of α-contractiveness. In [Gra94] the semi-metric given by $d'(t_1,t_2) = M|t_1 - t_2|$, $t_1, t_2 \in I$ are under consideration where M is some Lipschitz constant. So only Lipschitz continuous arcs would generate 0-contractive homotopies. M. Frigon [Fri96] considers more general semi-metrics like $d'(t_1,t_2) = |\phi(t_1) - \phi(t_2)|$, $t_1, t_2 \in I$ with some continuous function $\phi : I \longrightarrow \mathbb{R}$, see also [FGG95]. Observe that every continuous semi-metric d' on I generates such a function ϕ by $\phi(t) := d'(t, 0)$, $t \in I$. Hence, by the triangle inequality, our (3.5) is weaker than the corresponding condition in [Fri96].

For later purpose, see Corollary 3.4, we remark that restrictions and concatenations of α-contractive families are α-contractive families. In fact, for restrictions this is straightforwardly shown. For concatenations

$$h^t(x) := \begin{cases} f^{2t}(x), & 0 \leq t \leq \frac{1}{2} \\ g^{2t-1}(x), & \frac{1}{2} < t \leq 1, \end{cases} \qquad x \in A \tag{3.6}$$

of α-contractive families f^t and g^t, holding $f^1 = g^0$, we have to pay attention to (3.5). Let d'_f and d'_g be the continuous semi-metrics of (3.5) with respect to f^t and g^t, respectively. The functional $d' : I \times I \longrightarrow \mathbb{R}$ given by

$$d'_h(t,s) := \begin{cases} d'_f(2t, 2s), & 0 \leq t, s \leq \frac{1}{2} \\ d'_g(2t-1, 2s-1), & \frac{1}{2} \leq t, s \leq 1 \\ d'_f(2t, 1) + d'_g(2s-1, 0), & t \leq \frac{1}{2} \leq s \\ d'_f(2s, 1) + d'_g(2t-1, 0), & s \leq \frac{1}{2} \leq t \end{cases}$$

is a well-defined continuous semi-metric on I and provides (3.5) with respect to h^t.

Regularity requires additional assumptions on f^t:

Example 3.5. Let $f^t : A \longrightarrow X$ be an α-contractive family which is bounded, i. e. $\{f^t(x) \,;\, x \in A, t \in I\}$ is bounded. Suppose $Fix((f^t)) \cap \partial A = \emptyset$. Then f^t is a regular homotopy.

Proof. In [Gra94] it is shown that

$$J := \{t \in I \,;\, x^t = f^t(x^t) \text{ for some } x^t \in \mathring{A}\}$$

is closed and open provided $Fix((F^t)) \cap \partial A = \emptyset$. The proof of this fact carries through for our generalization of α-contractiveness, too. We omit details here since the proof of Proposition 5.1, see Chapter 5, treats a more general situation.

Hence $J = \emptyset$ or $J = I$. If $J = \emptyset$ we infer $A \neq X$ since otherwise $f^0 : X \longrightarrow X$ would have a fixed point $x^0 \in \overset{\circ}{X} = X$ by the Banach fixed point theorem, see Theorem 1.12. Hence, choosing $x_0 \in X \setminus A$,

$$\begin{tikzcd} A \arrow[rr, "f^t"] \arrow[dr, "x_0"'] \arrow[rr, phantom, shift right=4ex, "\gamma"] & & X \\ & \{x_0\} \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{3.7}$$

for any $\gamma \in \Gamma(X)$ which is approximability.

If $J = I$ observe that x^t is the unique fixed point of f^t and from α-contractiveness of f^t we infer continuity of $t \mapsto x^t$, i. e. $t \mapsto x^t$ is an arc in A. Approximability follows now from

$$\begin{tikzcd} A \arrow[rr, "f^t"] \arrow[dr, "t"'] \arrow[rr, phantom, shift right=4ex, "\gamma"] & & X \\ & I \arrow[ur, "x^t"'] & \end{tikzcd} \tag{3.8}$$

To prove the subnet condition let $x_{\gamma'} = \left(\psi_{\gamma'}^{t_{\gamma'}} \varphi_{\gamma'}^{t_{\gamma'}}\right)(x_{\gamma'})$ for some $(\psi_{\gamma'}^t, \varphi_{\gamma'}^t) \in (f^t, \gamma')$, $(V_{\gamma'}, X'_{\gamma'}) = \gamma' \in \Gamma' \leq \Gamma$. Fix $\varepsilon > 0$. Switching to a subnet of Γ', for economy of notation denoted by Γ', too, we can assume the existence of $\gamma'_0 \in \Gamma'$ such that $V_{\gamma'} \subseteq V_\varepsilon$ and $d'(t_{\gamma'_1}, t_{\gamma'_2}) \leq \varepsilon$ if $\gamma', \gamma'_1, \gamma'_2 \geq \gamma_0$. By $V_{\gamma'}$-proximity there are $y_{\gamma_1}, y_{\gamma_2} \in A$ such that $d(x_{\gamma_1}, y_{\gamma_1}) \leq \varepsilon$, $d(x_{\gamma_2}, y_{\gamma_2}) \leq \varepsilon$ and

$$\begin{aligned} d\left(x_{\gamma_1}, x_{\gamma_2}\right) &\leq 2\varepsilon + d\left(f^{t_{\gamma_1}}(y_{\gamma_1}), f^{t_{\gamma_2}}(y_{\gamma_2})\right) \\ &\leq 2\varepsilon + d\left(f^{t_{\gamma_1}}(y_{\gamma_1}), f^{t_{\gamma_1}}(y_{\gamma_2})\right) + d\left(f^{t_{\gamma_1}}(y_{\gamma_2}), f^{t_{\gamma_2}}(y_{\gamma_2})\right) \\ &\leq 2\varepsilon + \alpha d(y_{\gamma_1}, y_{\gamma_2}) + d'(t_{\gamma_1}, t_{\gamma_2}) \\ &\leq 3\varepsilon + \alpha\left(d(x_{\gamma_1}, x_{\gamma_2}) + 2\varepsilon\right). \end{aligned}$$

Again, since $(x_{\gamma'})_{\gamma' \geq \gamma_0}$ is bounded, as in the proof of Proposition 2.4, $(x_{\gamma'})_{\gamma'}$ has a subnet $(x_{\gamma''})_{\gamma''}$ which is Cauchy and converges by means of completeness of X. □

Obviously any map $F : A \longrightarrow 2^X$ defines a constant homotopy F^t by $F^t := F$, $t \in I$ and F is closed iff so is F^t. This equivalence holds for regular maps, too, i. e. we have generalized regularity in a natural way. More precise is

Proposition 3.1. *A map $F : A \longrightarrow 2^X$ is regular iff the constant homotopy $F^t = F$ is regular.*

Proof. As mentioned above F is closed iff so is F^t.

Fix $\gamma \in \Gamma$. It is straightforwardly shown that $(\psi, \varphi) \in (F, \gamma)$ implies $(\psi^t, \varphi^t) \in (F^t, \gamma)$ with constant homotopies $\psi^t := \psi$ and $\varphi^t := \varphi$, $t \in I$. Moreover for any fixed $t_0 \in I$ we infer $(\psi^{t_0}, \varphi^{t_0}) \in (F, \gamma)$ provided $(\psi^t, \varphi^t) \in (F^t, \gamma)$. Thus F is approximable iff so is F^t. From the above implications we moreover infer straightforwardly that F fulfills the subnet condition iff so does F^t. $\square$

As for regular maps we examine the construction of new regular homotopies from old ones.

Proposition 3.2. *Let $F^t : A \longrightarrow 2^X$ be a regular homotopy and $\lambda : I \longrightarrow I$ continuous. Suppose in addition X, A or F^t is compact or λ is a homeomorphism. Then $F^{\lambda(t)} : A \longrightarrow 2^X$ is a regular homotopy.*

Proof. Since F^t is closed and λ continuous $F^{\lambda(t)}$ is closed.

If $(\psi^t, \varphi^t) \in (F^t, \gamma)$ for some $\gamma \in \Gamma$ then $(\psi^{\lambda(t)}, \varphi^{\lambda(t)}) \in (F^{\lambda(t)}, \gamma)$. Hence approximability of F^t implies approximability of $F^{\lambda(t)}$.

If X or A is compact approximabilty of $F^{\lambda(t)}$ is equivalent with regularity and we are done. Moreover if F^t is compact so is $F^{\lambda(t)}$ and nothing remains to show.

Assume now λ is a homeomorphism. To prove the subnet condition for $F^{\lambda(t)}$ let $x_{\gamma'} = \big(\psi_{\gamma'}^{t_{\gamma'}} \varphi_{\gamma'}^{t_{\gamma'}}\big)(x_{\gamma'})$ for some $t_{\gamma'} \in I$ and $(\psi_{\gamma'}^t, \varphi_{\gamma'}^t) \in (F^{\lambda(t)}, \gamma')$ where $\gamma' \in \Gamma' \leq \Gamma$. Then $\big(\psi_{\gamma'}^{\lambda^{-1}(t)}, \varphi_{\gamma'}^{\lambda^{-1}(t)}\big) \in (F^t, \gamma')$ and $x_{\gamma'} = \big(\psi_{\gamma'}^{\lambda^{-1}(\lambda(t_{\gamma'})} \varphi_{\gamma'}^{\lambda^{-1}(\lambda(t_{\gamma'}))}\big))(x_{\gamma'})$ for all $\gamma' \in \Gamma'$. Since the homotopy F^t is regular we infer from its subnet condition the convergence of at least one subnet of $(x_{\gamma'})_{\gamma'}$ which was our aim. $\square$

Clearly the above compactness assumption mirrors them aim to obtain Proposition 3.2 for some quite general subclass of all regular homotopies without restrictions on λ. To postulate λ is a homeomorphism is - in some sense - the other end of the road. However, α-contractive families work well together with arbitrary continuous λ. We omit the straightforward proof of

Example 3.6. Let $f^t : A \longrightarrow X$ be a α-contractive family and $\lambda : I \longrightarrow I$ continuous. Then $f^{\lambda(t)}$ is a α-contractive family, too.

If f^t is in addition bounded and $Fix((f^t)) \cap \partial A = \emptyset$ then $f^{\lambda(t)}$ is a regular homotopy.

Proposition 3.3. *Let $F^t : A \longrightarrow 2^X$ be a regular homotopy and $\mu : A \longrightarrow I$ continuous. Suppose there exists $V_0 \in \mathcal{V}(X)$ and $t_0 \in I$ such that F^{t_0} is regular and*

$$\mu = t_0 \quad on \quad Fix((V_0 F^t)). \tag{3.9}$$

Then $F^\mu : A \longrightarrow 2^X$ given by $(F^\mu)(x) := F^{\mu(x)}(x)$, $x \in A$ is a regular map.

Proof. Since F^t is closed and μ continuous F^μ is closed.

Choose $V_1 \in \mathcal{V}(X)$ such that $V_1 V_1 \subseteq V_0$. We claim

$$Fix(V F^\mu V_A) = Fix(V F^{t_0} V_A), \quad V \in \mathcal{V}(X), \quad V \subseteq V_1. \tag{3.10}$$

Then, by Lemma 2.1, F^μ is regular since F^{t_0} is assumed to be.

To show (3.10) let $V \subseteq V_1$ and assume $x \in (VF^\mu V_A)(x)$. Then there exist $y \in A$, $z \in X$ such that $(x,y) \in V$, $(x,z) \in V$ and $z \in F^{\mu(y)}(y)$. In particular $y \in Fix(VVF^{\mu(y)})$ and hence y belongs to $Fix((V_0F^t))$ where $\mu = t_0$ holds. Thus $\mu(y) = t_0$, $z \in F^{t_0}(y)$ and $x \in Fix(VF^{t_0}V_A)$.

Assuming $x \in Fix(VF^{t_0}V_A)$ we infer in the same manner $x \in Fix(VF^\mu V_A)$. □

In presence of compactness the assumed regularity of F^{t_0} becomes redundant. Moreover we get rid of the uniform formulation. More precisely is

Corollary 3.1. *Let $F^t : A \longrightarrow 2^X$ be a compact regular homotopy and $\mu : A \longrightarrow I$ continuous. Suppose*

$$\mu \quad \text{is constant in a neighborhood of} \quad Fix((F^t)). \tag{3.11}$$

Then $F^\mu : A \longrightarrow 2^X$ is a regular map.

Proof. From Proposition 3.2 with $\lambda = t_0$ we infer regularity of F^{t_0}. Assumption (3.9) follows directly from (3.11) and compactness of F^t. Thus the suppositions of Proposition 3.3 are fulfilled and F^μ is regular. □

In view of our general philosophy of having no abscissa and ordinate (3.11) is more natural than (3.9) and comes back in (3.33) in a uniform formulation.

Consider now two regular homotopies $F^t : A \longrightarrow 2^X$, $G^t : A \longrightarrow 2^X$ such that $F^1 = G^0$. In general one cannot expect that their concatenation $(G^t)(F^t)$ given by

$$(G^t)(F^t)(x) := H^t(x) := \begin{cases} F^{2t}(x), & 0 \le t \le \frac{1}{2} \\ G^{2t-1}(x), & \frac{1}{2} \le t \le 1, \end{cases} \qquad x \in A \tag{3.12}$$

is regular, too.

Remark 3.2. In fact, the construction of a counterexample can be motivated by Theorem 2.6 for homotopies: the set of fixed points of a regular homotopy is arcwise connected in X.

Consider in $\mathbb{R}^2$ the compact pair (X, A) given by

$$\left(\{(-1,0),(1,0)\},\ (\{0\} \times [-1,1]) \cup \left\{ \left(x, \sin\frac{\pi}{x}\right)\ ;\ 0 < |x| \le 1 \right\} \right),$$

and the closed homotopies $F^t : A \longrightarrow 2^X$ and $G^t : A \longrightarrow 2^X$ given by the (set-valued) closed arcs

$$F^t := \begin{cases} \left(1-t, \sin\frac{\pi}{1-t}\right), & 0 \le t < 1 \\ \{0\} \times I, & t = 1 \end{cases}$$

and

$$G^t := \begin{cases} \left(-t, -\sin\frac{\pi}{t}\right), & 0 < t \le 1 \\ \{0\} \times I, & t = 0, \end{cases}$$

respectively.

Observe $Fix((F^t)) = Fix(F^0) = \{(1,0)\}$. Hence for any $\gamma \in \Gamma$

$$\begin{tikzcd} A \arrow[rr, "F^t"] \arrow[dr, "\star"'] \arrow[rr, phantom, shift right=4ex, "\gamma"] & & X \\ & \{\star\} \arrow[ur, "{(1-t,0)}"'] & \end{tikzcd}$$

which is approximability of F^t. Since X is compact F^t is regular.

From $Fix((G^t)) = Fix(G^1) = \{(-1,0)\}$ we infer regularity of G^t in an analogue manner.

Since $F^1 = \{0\} \times I = G^0$ the homotopy $(G^t)(F^t)$ is closed but fails to be regular in view of Theorem 2.6 for homotopies: $Fix(((G^t)(F^t))) = \{(-1,0),(1,0)\}$ and there is no arc from $(-1,0)$ to $(1,0)$ is X.

Proposition 3.5, see below, is a positive result how to connect regular homotopies.

In general we are forced to consider subclasses of the class of regular homotopies if there is need to connect homotopies. Examples 3.2, 3.3 and 3.5, are examples of such classes. Observe that the class of arcs, see Example 3.1, can be considered as such a class, too. In particular arcs are contained in most of the above classes.

However, for arbitrary regular homotopies F^t, G^t and H^t we infer from Proposition 3.2 straightforwardly that if $F^1 = G^0$ and $G^1 = H^0$ then $(H^t)((G^t)(F^t))$ is regular iff so is $((H^t)(G^t))(F^t)$.

In Proposition 3.5, see below, we examine a subclass of homotopies which preserves regularity under concatenation and which is more close to our general concept of approximation of the diagonal than to (classical) uniform approximation as in Examples 3.2, 3.3 and 3.5.

In view of (3.3) it is natural to examine regularity of $I \times F^t$ if F^t is a given regular homotopy.

Proposition 3.4. *Let $F^t : A \longrightarrow 2^X$ be regular and X, A or F^t compact. Then $F^I : I \times A \longrightarrow 2^{I\times X}$ given by*

$$F^I(t,x) := I \times F^t(x), \quad x \in A, \quad t \in I \tag{3.13}$$

is regular.

Proof. Since F^t is closed so is F^I. Thus, in view of the compactness assumption, it is sufficient to show approximability of F^I. Fix $\gamma \in \Gamma = \mathcal{A}(I \times A) \times \mathcal{V}(I \times X)$. There exists $\gamma' \geq \gamma$ such that $\gamma' = (I' \times X', V_\varepsilon \times_{(2,3)} V_X)$ where $I' \in \mathcal{A}(I)$, $X' \in \mathcal{A}(X)$, $\varepsilon > 0$, $V_X \in \mathcal{V}(X)$ and $\times_{(2,3)}$ is the cross-product with changed second and third coordinate. Choose $(\psi^t, \varphi^t) \in (F^t, (X', V_X))$.

By (3.3) the triangle

$$\begin{tikzcd} I \times A \arrow[rr, "F^I"] \arrow[dr, "(\Pi_I,\tilde{\varphi})"'] & \arrow[d, phantom, "\gamma'"] & I \times X \\ & I \times P \arrow[ur, "(\Pi_I,\psi)"'] & \end{tikzcd}$$

where Π_I denotes the projection onto I, is directly demonstrated. I. e. (F^I, γ') is non-empty. Thus (F^I, γ) is non-empty and we have shown approximability, hence regularity of F^I. □

Equivalence fails in Proposition 3.4.

Example 3.7. Consider $X = S^1$ the unit circle as a subset of the complex field. Let $f^t : S^1 \longrightarrow S^1$ be a family of rotations given by

$$f^t(e^{i\theta}) := e^{i(\theta+t)}, \quad \theta \in [0, 2\pi), \quad t \in I. \tag{3.14}$$

f^t is a compact continuous homotopy. We claim regularity of f^I given as in (3.13). Indeed, $I \times S^1$ is compact, $Fix(f^I) = \{0\} \times S^1$ is non-empty and any finite subset of the cylinder $I \times S^1$ can be connected by a simple arc. Hence, by Proposition 2.5, f^I is regular.

If f^t would be regular so would be f^1 by Proposition 3.2. This is impossible since f^1 is fixed point-free.

Remark 3.3. Example 3.7 shows that Proposition 3.2 would fail if one defines regularity for homotopies F^t by means of regularity of F^I which seems at a first glance and in view of (3.3) more natural than our definition.

3.2 A-regularity

We already mentioned that Theorem 2.5 is fundamental for this part of our considerations of regular maps and start this chapter with the definition of an appropriate term to handle (2.13).

Definition 3.4. Let (B, A) be a closed pair in X. A regular map $F : B \longrightarrow 2^X$ is said to be *A-regular* if F is uniformly upper semicontinuous on A and *A-approximable*, i. e. for each $(X', V) = \gamma \in \Gamma$ there exists $(\psi, \varphi) \in (F, \gamma)$ such that

$$\mathcal{G}(\psi\varphi|_A) \subseteq \mathcal{G}(VFV_A). \tag{3.15}$$

Let $I' \in \mathcal{A}(I)$. A regular homotopy is said to be *A-regular on I'* if for each $t' \in I'$ the map $F^{t'}$ is uniformly upper semicontinuous on A and the homotopy F^t is *A-approximable on I'*, i. e. for each $(X', V) = \gamma \in \Gamma(X) = \mathcal{A}(B) \times \mathcal{V}(X)$ there exists $(\psi^t, \varphi^t) \in (F^t, \gamma)$ such that

$$\mathcal{G}(\psi^{t'}\varphi^{t'}|_A) \subseteq \mathcal{G}(VF^{t'}V_A), \qquad t' \in I'. \tag{3.16}$$

A homotopy F^t is said to be *A-approximable* (*A-regular*) if F^t is A-approximable (A-regular) on each $I' \in \mathcal{A}(I)$.

Hence A-regularity of a map $F : B \longrightarrow 2^X$ means that $(\psi\varphi)|_A$ approximates the (uniformly upper semicontinuous) map $F|_A$ uniformly. Observe that every regular map $F : A \longrightarrow 2^X$ is a $\emptyset$-regular map.

Let us agree to use the notation *A-approximable (A-regular) in $t = t'$* if the above $I' \in \mathcal{A}(I)$ is a singleton $I' = \{t'\}$.

Proposition 2.3 generalizes immediately to

Example 3.8. All constant functions $x_0 : A \longrightarrow X$ are A-regular.

The approximations of Examples 3.1, 3.2 and 3.3 are uniform, too. Hence these examples generalize to

Example 3.9. Any arc $c : I \longrightarrow X$ defines an A-regular homotopy $c^t : A \longrightarrow X$ as in Example 3.1.

Example 3.10. Let E be a Fréchet space and $F^t : A \longrightarrow 2^E$ closed and compact with convex values.

Then F^t is A-approximable. Moreover F^t is on every $I' \in \mathcal{A}(I)$ A-regular where for each $t' \in I'$ $F^{t'}$ is uniformly upper semicontinuous.

Example 3.11. Let X be metric and AE(*compact metric*). (E. g. X is AR (*compact metric*).) Then any compact continuous homotopy $f^t : A \longrightarrow X$ is A-approximable. Moreover f^t is on every $I' \in \mathcal{A}(I)$ A-regular where for each $t' \in I'$ $f^{t'}$ is uniformly semicontinuous.

Moreover Example 3.5 generalizes to

Example 3.12. Let (X, d) be a complete metric space and $f^t : A \longrightarrow X$ a bounded α-contractive family such that $Fix((f^t)) \cap \partial A = \emptyset$. Then f^t is on every $I' \in \mathcal{A}(I)$ A-regular where for each $t' \in I'$ $f^{t'}$ is a constant $x_{t'}$.

Indeed, regularity follows from Example 3.5, the $f^{t'}$ are uniformly upper semicontinuous as constant functions and A-approximablility follows directly from the proof of Example 3.5: If there exists some $t' \in I'$ such that $f^{t'} = x_{t'} \in X \setminus A$ then $(\iota, x_{t'}) \in (f^t, \gamma)$, $\gamma \in \Gamma$ by (3.7), if $f^{t'} = x_{t'} \in A$ for some $t' \in I'$ then $(x^t, t) \in (f^t, \gamma)$, $\gamma \in \Gamma$ by (3.8).

At the end of this chapter we make use of A-regularity to examine a subclass of the class of regular homotopies which preserves concatenation. C. f. Remark 3.2.

Proposition 3.5. *Let X be a contractible and locally contractible metric space. Let $F^t : A \longrightarrow 2^X$ and $G^t : A \longrightarrow 2^X$ be A-regular in $t_0 = 1$ and $t_0 = 0$, respectively, and suppose $F^1 = x_0 = G^0$.*

If $x_0 \in A$ assume in addition that the associated maps $\tilde{F}$ and $\tilde{G}$ are upper semicontinuous in $(1, x_0)$ and $(0, x_0)$ respectively.

Then the homotopy $H^t = (G^t)(F^t)$, given by (3.12), is A-regular in $t_0 = \frac{1}{2}$.

Proof. Apparently H^t is a closed homotopy and $H^{\frac{1}{2}}$ is uniformly upper semicontinuous.

We start to show regularity of H^t. Fix $(X', V) = \gamma \in \Gamma(X)$. We claim the existence of a triangle

$$\begin{tikzcd} A \arrow[rr, "H^t"] \arrow[dr, "\varphi_H^t"'] & \gamma & X \\ & P_H \arrow[ur, "\psi_H^t"'] & \end{tikzcd} \tag{3.17}$$

Without loss of generality we assume $x_0 \in X'$ and consider first the case $x_0 \in A$.

Since $\tilde{F}$ is upper semicontinuous in $(1, x_0)$, $\tilde{G}$ in $(0, x_0)$ and $F^1 = x_0 = G^0$ there exist $V' \in \mathcal{V}(X)$ and $\delta > 0$ such that

$$\tilde{F}([1-\delta, 1] \times V'_A(x_0)) \subseteq V(x_0) \qquad \text{and} \qquad \tilde{G}([0, \delta] \times V'_A(x_0)) \subseteq V(x_0). \tag{3.18}$$

Choose a closed $V'' \in \mathcal{V}(X)$ such that $V'' \subseteq V'$ and Lemma 1.2 holds with respect to $O = V'(x_0)$ and $Q = V''(x_0)$. Choose

$$(\psi_F^t, \varphi_F^t) \in (F^t, (X', \mathring{V}'')) \qquad \text{and} \qquad (\psi_G^t, \varphi_G^t) \in (G^t, (X', \mathring{V}'')) \tag{3.19}$$

such that, from A-approximability,

$$\mathcal{G}(\psi_F^1 \varphi_F^1) \subseteq \mathcal{G}(\mathring{V}''(x_0)_A) \qquad \text{and} \qquad \mathcal{G}(\psi_G^0 \varphi_G^0) \subseteq \mathcal{G}(\mathring{V}''(x_0)_A). \tag{3.20}$$

Let P_F and P_G be the convex polyhedra, associated to (ψ_F^t, φ_F^t) and (ψ_G^t, φ_G^t), respectively, and $\tilde{\psi}_F$, $\tilde{\varphi}_F$, $\tilde{\psi}_G$ and $\tilde{\varphi}_G$ the associated continuous functions.

Since $x_0 \in X'$ and $x_0 \in A$

$$x_0 = (\psi_F^1 \varphi_F^1)(x_0) = (\psi_G^0 \varphi_G^0)(x_0)$$

by preciseness. Thus, by continuity of $\tilde{\varphi}_F$, $\tilde{\psi}_G$ and $\tilde{\varphi}_G$, there exists $0 < \delta' \leq \delta$ such that

$$(\psi_F^t \varphi_F^t)(x_0) \in \mathring{V}''(x_0), \quad t \in [1-\delta', 1] \tag{3.21}$$

and

$$(\psi_G^t \varphi_G^t)(x_0) \in \mathring{V}''(x_0), \quad t \in [1, \delta']. \tag{3.22}$$

Set

$$\tilde{P}_F := (\tilde{\psi}_F)^{-1}(V''(x_0)) \quad \text{and} \quad \tilde{P}_G := (\tilde{\psi}_G)^{-1}(V''(x_0)). \tag{3.23}$$

By (3.21) to (3.23) $\tilde{P}_F$ and $\tilde{P}_G$ are closed neighborhoods of

$$\begin{aligned} \tilde{P}'_F &:= \tilde{\psi}_F^{-1}(x_0) \cap ([1-\delta', 1] \times \tilde{\varphi}_F([1-\delta', 1] \times \{x_0\})) \quad \text{and} \\ \tilde{P}'_G &:= \tilde{\psi}_G^{-1}(x_0) \cap ([0, \delta'] \times \tilde{\varphi}_G([0, \delta'] \times \{x_0\})) \end{aligned} \tag{3.24}$$

in $I\times P_F$ and $I\times P_G$, respectively. Consider $\tilde{P}_F$ and $cone\,\tilde{P}'_F$ as subsets of $cone\,\tilde{P}_F$ and define $\tilde{\psi}'_F : \tilde{P}_F \cup cone(\tilde{P}'_F) \longrightarrow V''(x_0)$ by

$$\tilde{\psi}'_F = \tilde{\psi}_F \quad \text{on} \quad \tilde{P}_F \qquad \text{and} \qquad \tilde{\psi}'_F = x_0 \quad \text{on} \quad cone(\tilde{P}'_F) \tag{3.25}$$

which is well-defined by (3.24). By Lemma 1.2 and the choice of V'' $\tilde{\psi}'_F$ admits a continuous extension to $\tilde{\psi}''_F : cone\,\tilde{P}_F \longrightarrow V'(x_0)$.

Analoguously we infer an extension $\tilde{\psi}''_G : cone\,\tilde{P}_G \longrightarrow V'(x_0)$ of $\tilde{\psi}'_G = \tilde{\psi}_G$ on $\tilde{P}_G$ and $\tilde{\psi}'_G = x_0$ on $cone\tilde{P}'_F$.

There are embeddings

$$cone\,\tilde{P}_F \quad \hookrightarrow \quad \left[0, \frac{1}{2}\right] \times (P_F \star \{*\}) \quad \hookrightarrow \quad I \times (P_F \star P_G)\,.$$

Using notation (1.1), the first one is given by

$$((1-s)(t,p) + s*) \mapsto \left((1-s)\frac{t}{2} + s\frac{1}{2}, (1-s)p + s*\right)$$

for $(t,p) \in \tilde{P}_F$. The second one is affine and comes from identification of the tip $*$ with $(\frac{1}{2}p_F + \frac{1}{2}p_G)$ which is an element of $P_F \star P_G$. This is possible since P_F is convex.

Analoguously there are embeddings

$$cone\,\tilde{P}_G \quad \hookrightarrow \quad \left[\frac{1}{2}, 1\right] \times (\{*\} \star P_G) \quad \hookrightarrow \quad I \times (P_F \star P_G)\,,$$

where the first one is given by

$$((1-s)(t,p) + s*) \mapsto \left((1-s)\frac{1}{2} + s\frac{1+t}{2}, (1-s)*+sp\right)$$

for $(t,p) \in \tilde{P}_G$.

Moreover both $I \times P_F$ and $I \times P_G$ embed into $I \times (P_F \star P_G)$ by $(t,p_1) \mapsto (\frac{t}{2}, 1p_1 + 0p_2)$ and $(t,p_1) \mapsto (\frac{1+t}{2}, 0p_1 + 1p_2)$, respectively.

In what follows we identify the spaces $cone\,\tilde{P}_F$, $cone\,\tilde{P}_G$, $I \times P_F$ and $I \times G$ with their related subspaces of $I \times (P_F \star P_G)$.

Consider their union

$$\tilde{P}'' := (I \times P_F) \cup (I \times P_G) \cup (cone\,\tilde{P}_F) \cup (cone\,\tilde{P}_G) \tag{3.26}$$

in $I \times (P_F \star P_G)$ and define $\tilde{\psi}''_H : \tilde{P}'' \longrightarrow X$ by

$$\tilde{\psi}''_H(t,x) := \begin{cases} \tilde{\psi}_F(t,x), & (t,x) \in (I \times P_F) \setminus (cone\,\tilde{P}_F), \\ \tilde{\psi}''_F(t,x), & (t,x) \in cone\,\tilde{P}_F \end{cases} \tag{3.27}$$

if $0 \leq t \leq \frac{1}{2}$ and use $\tilde{\psi}_G$ and $\tilde{\psi}''_G$ instead of $\tilde{\psi}_F$ and $\tilde{\psi}''_F$, respectively to define $\tilde{\psi}''_H(t,x)$ if $\frac{1}{2} \leq t \leq 1$. By definition of $\tilde{\psi}'_F$, see (3.25), and the analogue formulation for ψ'_G, $\tilde{\psi}''_H$ is well-defined, continuous and admits a continuous extension $\tilde{\psi}_H : I \times (P_F \star P_G) \longrightarrow X$ by Lemma 1.4.

Define $P_H := P_F \star P_G$ and let ψ^t_H be the homotopy associated with $\tilde{\psi}_H$.

We switch to the definition of φ^t_H.

For each $x' \in X'$ let

$$J_F(x') := \{t \in I\,;\, x' \in F^t(x')\} \qquad \text{and} \qquad J_G(x') := \{t \in I\,;\, x' \in G^t(x')\}.$$

All $J_F(x')$ are closed and from $F^1 = x_0$ we infer $1 \in J_F(x')$ iff $x' = x_0$. Since $I \times A$ is normal there exist a closed neighborhood U_F of $(\{1\} \times A)$ disjoint to $\cup_{x' \in X' \setminus \{x_0\}}(J_F(x') \times \{x'\})$. Let $\Pi_I : I \times A \longrightarrow I$ denote the projection onto I factor. By (3.20), (3.23)

$$\tilde{A}_F := ([1-\delta', 1] \times A) \cap U_F \cap (\Pi_I, \tilde{\varphi}_F)^{-1}(\tilde{P}_F) \tag{3.28}$$

is a neighborhood of $\{1\} \times A$. Choose an Urysohn function $\mu_F : I \times A \longrightarrow I$ vanishing on $(I \times A) \setminus \tilde{A}_F$ and equal to 1 on $\{1\} \times A$. Since P_F is convex we can embed *cone* P_F, as above, affinely into $P_F \star P_G$ by means of embedding P_F and identification of the tip with $(\frac{1}{2}p_F + \frac{1}{2}p_G)$.

From an analogue argumentation we infer an Urysohn function $\mu_G : I \times A \longrightarrow I$ which vanishes on $(I \times A) \setminus \tilde{A}_G$ where $\tilde{A}_G = ([0,\delta'] \times A) \cap U_G \cap (\Pi_I, \tilde{\varphi}_G)^{-1}(\tilde{P}_G)$ and U_G is a closed neighborhood of $\{0\} \times A$ disjoint to $\cup_{x' \in X' \setminus \{x_0\}}(J_G(x') \times \{x'\})$, and which is equal to 1 on $\{0\} \times A$. Also embed *cone* P_G into $P_F \star P_G$.

Define $\tilde{\varphi}_H : I \times A \longrightarrow P_F \star P_G$ by

$$\tilde{\varphi}_H(t,x) := \begin{cases} (1-\mu_F(2t,x))\tilde{\varphi}_F(2t,x) + \mu_F(2t,x)\left(\frac{1}{2}p_F + \frac{1}{2}p_G\right), \\ \hfill (t,x) \in \left[0, \frac{1}{2}\right] \times A \\ (1-\mu_F(2t-1,x))\tilde{\varphi}_G(2t-1,x) + \mu_G(2t-1,x)\left(\frac{1}{2}p_F + \frac{1}{2}p_G\right), \\ \hfill (t,x) \in \left[\frac{1}{2}, 1\right] \times A \end{cases} \tag{3.29}$$

and by means of the above embeddings. Let φ^t_H be the homotopy associated with $\tilde{\varphi}_H$.

We claim

$$\begin{tikzcd} A \arrow[rr, "H^t"] \arrow[dr, "\varphi^t_H"'] & (X', VV) & X \\ & P_F \star P_G \arrow[ur, "\psi^t_H"'] & \end{tikzcd} \tag{3.30}$$

To see preciseness with respect to X' let $x' \in X' \cap H^{t'}(x')$ for some $t' \in I$. If $0 \leq t' \leq \frac{1}{2}$ we infer $x' \in X' \cap F^{2t'}(x')$ hence

$$x' = \left(\psi^{2t'}_F \varphi^{2t'}_F\right)(x') \tag{3.31}$$

by (3.19) and preciseness of (ψ^t_F, φ^t_F) with respect to X'.

If $\mu_F(2t', x') = 0$ then

$$\left(\psi^{t'}_H \varphi^{t'}_H\right)(x') = \left(\psi^{t'}_H\left(\tilde\varphi_F(2t', x')\right)\right) = \left(\psi^{2t'}_F \varphi^{2t'}_F\right)(x') = x'$$

by (3.29), (3.27) ($\tilde\varphi_F$ maps to P_F) and (3.31).

If $\mu_F(2t', x') > 0$ then $x' = x_0$ by the choice of U_F and, using notation (1.1),

$$\left(\psi^{t'}_H \varphi^{t'}_H\right)(x_0) \in \psi^{t'}_H\left(\left[\tilde\varphi_F(2t', x_0), \frac{1}{2}p_F + \frac{1}{2}p_G\right]\right) \tag{3.32}$$

by (3.29) and (3.28). From (3.31) we infer $\tilde\psi_F(2t', \tilde\varphi_F(2t', x_0)) = x_0$, i. e. $(2t', \tilde\varphi_F(2t', x_0)) \in \tilde\psi_F^{-1}(x_0)$. Since (3.28) and $\mu_F(2t', x_0) > 0$ imply also $2t' \in [1-\delta', 1]$ we infer $(2t', \tilde\varphi_F(2t', x_0)) \in [1-\delta', 1] \times \tilde\varphi_F([1-\delta', 1] \times \{x_0\})$ and, by (3.24), $(2t', \tilde\varphi_F(2t', x_0)) \in \tilde P'_F$. Thus, by (3.27) and (3.25), the right hand side of (3.32) is the singleton $\{x_0\}$ and we have shown preciseness with respect to $\{x_0\}$.

We have shown preciseness of (3.17) for the case $x_0 \in A$.

If $\frac{1}{2} < t' \le 1$ we can argue in the same manner using preciseness of G^t with respect to X'.

To see (VV)-proximity of (ψ^t_H, φ^t_H) with respect to H^t we again consider only the case $0 \le t \le \frac{1}{2}$. Let $x = (\psi^t_H \varphi^t_H)(x)$. If $\mu_F(2t, x) = 0$ then $x \in Fix(VH^tV_A)$ follows straightforwardly from (VV)-proximity of (ψ^t_F, φ^t_F) with respect to F^t along the lines of the above proof of preciseness .

If $\mu(2t, x) > 0$ then $x = (\psi^t_H \varphi^t_H)(x) \in \tilde\psi''_F(cone\, \tilde P_F)$ by (3.28), (3.29), hence $x \in V'(x_0)$ since $\tilde\psi''_F$ maps to $V'(x_0)$. Moreover, as in the proof of preciseness, we infer $2t \in [1-\delta', 1]$, thus $(2t, x) \in [1-\delta, 1] \times V'_A(x_0)$ since $\delta \ge \delta'$ and $F^{2t}(x) \subseteq V(x_0)$ by (3.18). Hence $x \in V_A(x_0) \subseteq (VVF^{2t})(x) \subseteq (VVF^{2t}V_A)(x)$ and $x \in Fix((VVF^{2t}(VV)_A)$.

We have shown proximity of (3.17) for the case $x_0 \in A$.

The case $x_0 \notin A$ is less complicated and does not make use of the additional continuity assumption on $\tilde F$ and $\tilde G$ in $(1, x_0)$ and $(0, x_0)$, respectively. In fact, since x_0 can be separated from A we can make use of the above construction of ψ^t_H and φ^t_H, omitting the parts where we took care of the fixed point x_0.

We already mentioned at the beginning of the proof that $H^{\frac{1}{2}}$ is uniformly upper semicontinuous. Hence we infer A-approximability in $t = \frac{1}{2}$ since by the above construction of $\left(\psi^{1/2}_H \varphi^{1/2}_H\right) = x_0$.

The subnet condition proves directly from the subnet conditions of F^t and G^t, respectively, and from the observation that for any $(\psi^t, \varphi^t) \in (H^t, \gamma)$ we infer $(\psi^{t/2}, \varphi^{t/2}) \in (F^t, \gamma)$ and $(\psi^{(1+t)/2}, \varphi^{(1+t)/2}) \in (G^t, \gamma)$. □

Observe that, by a separation argument, it is moreover possible to show the preservation of A-regularity of F^t in $t_0 = 0$ and of G^t in $t = 1$ in common, i. e. if F^t and G^t are A-regular in $I' = \{0, 1\}$ so is H^t.

3.3 Topological Transversality

Definition 3.5. Let (B, A) be a closed pair in X. A closed map $F : A \longrightarrow 2^X$ is said to be *inessential* with respect to B if there exists a fixed point-free A-regular extension $\hat{F} : B \longrightarrow 2^X$ of F. F is said to be *essential* (or *traverse*) with respect to B if it is not inessential with respect to B.

Theorem 3.1. *Let X be a contractible and locally contractible metrizable space, $Q \subseteq X$ non-empty and closed, R a union of components of $X \setminus Q$ and $x_0 \in X \setminus Q$. Then the constant function $x_0 : Q \longrightarrow X$ is essential with respect to $Q \cup R$ iff $x_0 \in R$.*

Proof. If $x_0 \notin R$ then

$$\begin{tikzcd} Q \cup R \arrow[rr, "x_0"] \arrow[dr, "x_0"'] & \gamma & X \\ & \{x_0\} \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

for all $\gamma \in \Gamma$. Hence $x_0 : Q \cup R \longrightarrow X$ is a fixed point-free Q-regular extension of $x_0 : Q \longrightarrow X$.

Assume now $x_0 \in R$ and there exists a fixed point-free Q-regular extension $F : Q \cup R \longrightarrow 2^X$ of $x_0 : Q \longrightarrow X$.

We claim F fulfills the assumptions of Theorem 2.5. Indeed, X is a contractible and locally contractible metric space, x_0 is an inner point of R, hence of $A := Q \cup R$. Since F is Q-regular it is regular. Since $\partial A \subseteq Q$ F is also uniformly upper semicontinuous of ∂A and holds (2.13). Thus $\hat{F} : X \longrightarrow 2^X$, given by (2.14), is a regular extension of F. Since $\hat{F}$ is apparently fixed point-free we infer a contradiction. $\square$

Theorem 3.2 (Homotopy Extension Theorem). *Let (B, A) be a closed pair in X such that B is normal. Let $F^t : B \longrightarrow 2^X$ be a regular homotopy. Suppose there exists $V_0 \in \mathcal{V}(X)$ such that*

$$A \cap Fix((V_0 F^t V_{0B})) = \emptyset. \tag{3.33}$$

Suppose F^0 is regular, fixed point-free and F^t is A-regular in $t = 1$. Then $F^1|_A$ is inessential with respect to B.

Proof. Choose $V_1 \in \mathcal{V}(X)$ such that $V_1 V_1 V_1 \subseteq V_0$. $C := \overline{Fix((V_1 F^t V_{1B}))}$ is a closed neighborhood of $Fix((F^t))$ in B. By (3.33) C is disjoint to $D := \overline{V_{1B}(A)}$. Since B is normal there exists an Urysohn function $\mu : B \longrightarrow I$ vanishing on C and equal to 1 on D.

We claim $F^\mu : B \longrightarrow 2^X$ is a fixed point-free A-regular extension of $F^1|_A$.

Since μ is continuous and F^t closed F^μ is closed, too. (3.9) holds with respect to $t_0 = 0$ and $V_0 = V_1$ since $\mu = 0$ on C. F^0 is assumed to be regular. Thus, by Proposition 3.3, F^μ is regular.

Since $\mu = 1$ on A F^μ is an extension of $F^1|_A$. Furthermore any fixed point x of F^μ would belong to $Fix((F^t))$ where $\mu = 0$ holds. Hence x would be a fixed point of F^0, too, which is assumed to be fixed point-free. I. e. F^μ is fixed point-free.

It remains to demonstrate A-regularity of F^μ.

Since $\mu = 1$ on the uniform neighborhood D of A and since F^t is A-regular in $t = 1$ F^μ is uniformly upper semicontinuous on A.

To prove A-approximability of F^μ let $(X', V) = \gamma \in \Gamma$.

Since F^0 is regular and fixed point-free we can assume

$$Fix(\psi_\gamma \varphi_\gamma) = \emptyset \quad \text{for each} \quad (\psi_\gamma, \varphi_\gamma) \in (F^0, \gamma). \tag{3.34}$$

without loss of generality by Proposition 2.6. Moreover we can assume $V \subseteq V_1$.

Since F^t is A-approximable in $t = 1$ there exists $(\psi^t, \varphi^t) \in (F^t, \gamma)$ such that

$$\mathcal{G}(\psi^1 \varphi^1|_A) \subseteq \mathcal{G}(V F^1 V_A). \tag{3.35}$$

We claim

$$\begin{tikzcd} B \arrow[rr, "F^\mu"] \arrow[dr, "(\varphi^\mu, \mu)"'] & \gamma & X \\ & P \times I \arrow[ur, "\psi^{\Pi_I} \Pi_X"'] & \end{tikzcd} \tag{3.36}$$

where Π_I and Π_X are the projections onto I and X, respectively

Since P is a convex polyhedron so is $P \times I$. Define short hand for $\hat{\varphi} := (\varphi^\mu, \mu)$ and $\hat{\psi} := \psi^{\Pi_I} \Pi_X$ which are apparently continuous. Since F^μ is fixed point-free preciseness is redundant and to demonstrate proximity it is sufficient to show that $\hat{\psi}\hat{\varphi}$ is fixed point-free.

By (3.34) it is sufficient to show $(\hat{\psi}, \hat{\varphi}) \in (F^0, \gamma)$. Assume x is a fixed point of $\hat{\psi}\hat{\varphi}$, i. e. $x = \left(\psi^{\mu(x)} \varphi^{\mu(x)}\right)(x)$. Since $(\psi^t, \varphi^t) \in (F^t, \gamma)$ this implies $x \in Fix((V F^t V_B))$ by V-proximity of F^t. Since $V \subseteq V_1$ $Fix((V F^t V_B))$ is contained in C where μ vanishes and we infer $x = (\psi^0 \varphi^0)(x)$. Therefore by V-proximity of F^t $Fix(\hat{\psi}\hat{\varphi}) \subseteq Fix(V F^0 V_B)$, i. e. $(\hat{\psi}, \hat{\varphi}) \in (F^0, \gamma)$ since F^0 is fixed point-free and preciseness, therefore, redundant. We have shown (3.36).

We claim the uniform approximation property (3.15) for $F = F^\mu$ and $(\psi, \varphi) = (\hat{\psi}, \hat{\varphi})$. In fact, since $\mu = 1$ in the uniform neighborhood $D = \overline{V_1(A)}$ of A and $V \subseteq V_1$ (3.15) follows from (3.35).

We have shown A-approximability of F^μ. $\square$

Remark 3.4. We point out the difference between Theorem 3.2 and classical Homotopy Extension Theorems.

Starting with the work of K. Borsuk [Bor36] and continued by A. Granas [Gra59], V. Klee [Kle60a], S. Hahn [Hah77] and others the concept of Topological Transversality has been extensively investigated. K. Borsuk's originally used technique of extending homotopies becomes less important and the name *Homotopy Extension Theorem* was merely kept for historical reasons. See e. g. [Jer82, Chapter 2] for some historical background. One of the most general concepts in this direction is due to T. Jerofsky who considered classes of regular (in his sense) homotopies and maps which, basically, suppose their well behavior with respect to Urysohn functions, see [Jer82, Chapter 2.,2.1.10 Definition]. In particular for each homotopy $F^t : A \longrightarrow 2^X$ belonging to a fixed class of regular homotopies the map $F^\mu : A \longrightarrow 2^X$ is supposed to belong to the associated class of regular maps whenever μ is a continuous functional.

This is in contrast to our Corollary 3.1 where precisely for those Urysohn functions μ, which are needed for the proof of our Homotopy Extension Theorem, assumption (3.11) turns out to be naturally fulfilled and regular homotopies $F^t : A \longrightarrow 2^X$ generate regular maps $F^\mu : A \longrightarrow 2^X$.

3.4 Nonlinear Alternative and Sweeping Theorem

There are various consequences of the Homotopy Extension Theorem whenever some essential maps are known. We refer to the monograph [Gra62] and [DG82, §4] for an outline regarding continuous homotopies in normed spaces.

In view of Theorem 3.1 we are on the one hand interested in applications which can be formulated in more or less arbitrary contractible, locally contractible and metrizable spaces. On the other hand we are forced to consider statements which relie on (in)essentiality of constant maps. For these reasons we consider the Nonlinear Alternative and the Sweeping theorem.

Theorem 3.3 (Nonlinear Alternative). *Let X be a contractible and locally contractible metrizable space. Let $F^t : A \longrightarrow 2^X$ be a regular homotopy such that F^t is (∂A)-regular in $t = 1$. Suppose that $F^1|_{\partial A} = x_0$ for some $x_0 \in \mathring{A}$ and that F^0 is regular. Then for every $V \in \mathcal{V}(X)$ at least one of the following statements holds:*

(i) VF^0V_A has a fixed point.

(ii) There exists $0 < t < 1$ such that VF^tV_A has a fixed point on ∂A.

Proof. Assume to the contrary there exists $V_0 \in \mathcal{V}(X)$ such that neither (i) nor (ii) holds. We claim that the hypothesis of the Homotopy Extension Theorem, Theorem 3.2 are fulfilled with respect to the closed pair $(A, \partial A)$.

Indeed, A as a metric space, is normal. $\partial A \cap Fix((V_0F^tV_{0_A})) = \emptyset$ for all $0 \leq t < 1$ and $Fix(V_0F^0V_{0_A}) = \emptyset$ hold since neither (i) nor (ii) holds. Since F^t is (∂A)-regular in $t = 1$ the map F^1 is uniformly upper semicontinuous on ∂A. Thus, since $F^1|_{\partial A} = x_0$ and $x_0 \in \mathring{A}$, also $\partial A \cap Fix(F^1) = \emptyset$.

Moreover F^t and F^0 are regular, hence $F^1|_{\partial A} = x_0$ is inessential with respect to A by Theorem 3.2.

This contradicts Theorem 3.1 since $x_0 \in \mathring{A}$. □

In presence of compactness or for a bounded α-contractive family we obtain classical versions of the Nonlinear Alternative. Indeed, by Theorem 3.3 either (i) (or (ii)) holds for all V' belonging to some cofinal $\mathcal{V}'(X) \leq \mathcal{V}(X)$.

If F^t is a compact homotopy the fixed points $x_{V'}$ in (i) (in (ii)) have a cluster point which is apparently a fixed point of F^0 (of $F^t|_{\partial A}$ for some $0 < t < 1$).

If we are concerned with a bounded α-contractive family $F^t = f^t$ the fixed points $x_{V'}$ define a net which is Cauchy - recall the proofs of Proposition 2.4 and Example 3.5.

This results in

Corollary 3.2. *In addition to the hypothesis of Theorem 3.3 suppose that*

(a) F^t *is compact or*

(b) X *is complete and* $F^t = f^t$ *a bounded* α*-contractive homotopy.*

Then at least one of the following statements holds:

(i) F^0 *has a fixed point.*

(ii) There exists $0 < t < 1$ *such that* F^t *has a fixed point on* ∂A.

Recent developments characterize operators and spaces in terms of the statement of the Nonlinear Alternative, see [Gra93] and [HL01], respectively.

Our second result is the Sweeping theorem. It makes use of *fields*, i. e. the maps under consideration look like $id - F$ with some additional feature of F (e. g. compactness). Therefore X must at least have the algebraic structure of a group and for what follows we consider Hausdorff *topological groups* $(X, +)$, i. e. summation and building the inverse are continuous with respect to a given Hausdorff topology on X. As for HTVS this topology turns out to be translation invariant and corresponds to a unique uniformity $\mathcal{V}(X)$ on X.

For the proof of the Sweeping Theorem we have to consider sums and concatenations of regular homotopies. It is straightforwardly shown that for every compact topological group where summation of regular single-valued homotopies results in regular homotopies we infer that regularity is preserved with respect to concatenation of single-valued homotopies, too. In fact, if $f^t : A \longrightarrow X$ and $f^t : A \longrightarrow X$ are regular single-valued homotopies such that $f^1 = g^0$ then, by Proposition 3.2, the homotopies $f^{\min\{2t,1\}}$, $g^{\max\{0,2t-1\}}$ and g^0 are regular, too. Hence $(g^t)(f^t) = f^{\min\{2t,1\}} + g^{\max\{0,2t-1\}} - g^0$ is regular.

However, in general regularity fails to be preserved by summation. E. g. from Example 2.8 we know that $id : S^1 \longrightarrow S^1$, the identity on the sphere, is regular but summation with an - also regular - constant function a on S^1 yields a regular function iff $a \equiv 0 \mod 2\pi$. Therefore the formulation of the Sweeping Theorem requires assumptions regarding concatenation.

Theorem 3.4 (Sweeping Theorem). *Let $(X,+)$ be a contractible locally contractible and metrizable topological group, $A \subseteq X$ closed and non-empty, $\{a,b\} \subseteq X \setminus A$ and B the component of $X \setminus A$ which contains b.*

Let $F^t : X \longrightarrow 2^X$ be a closed homotopy such that $F^0 = 0$.

Suppose in addition that for each arc $c : I \longrightarrow X$, joining a and b, the restriction of the (closed) homotopy

$$H^t := (F^{1-t} + b)(F^1 + c(t))(F^t + a) \tag{3.37}$$

to $A \cup B$ is A-regular in $t = 1$.

Then we infer for every $V \in \mathcal{V}(X)$: if a, b belong to the same component of $X \setminus (id - VF^1V)(A)$ and to different components of $X \setminus V(A)$ then $\{a,b\}$ is swept by $(id - VF^tV)(A)$, i. e. there exists $t \in I$ such that

$$\{a,b\} \cap (id - VF^tV)(A) \neq \emptyset.$$

Proof. Observe first that by our discussion following Remark 3.2 regularity of (3.37) is a well-defined assumption. Moreover in what follows the order of concatenation in (3.37) will not be of interest.

Assume to the contrary that there exists $V \in \mathcal{V}(X)$ such that $\{a,b\}$ is not swept, i. e.

$$\{a,b\} \cap (id - VF^tV)(A) = \emptyset, \qquad t \in I. \tag{3.38}$$

Since X is locally contractible it is locally arcwise connected and so is M, the component of $X \setminus (id - VF^1V)(A)$ which contains a and b. Hence M is arcwise connected and there exists an arc $c : I \longrightarrow M$ from a to b.

Consider the homotopy $H^t : X \longrightarrow 2^X$ from (3.37) where c is the above arc in M. We claim

$$A \cap Fix((VH^tV)) = \emptyset. \tag{3.39}$$

In fact, $x \in Fix((VH^tV))$ iff $x \in Fix((V(F^{1-t}+b)V)) \cup Fix((V(F^1+c(t))V)) \cup Fix((V(F^t+a)V))$. This union equals $Fix((VF^tV+b)) \cup Fix((VF^1V+c(t))) \cup Fix((VF^tV+a))$ which is disjoint to A by (3.38) and since the arc c maps to M. Particularly we infer $A \cap Fix((VH^tV_{A\cup B})) = \emptyset$, i. e. (3.33) holds for the restricted homotopy $H^t|_{A\cup B} : A \cup B \longrightarrow 2^X$ which is assumed to be A-regular in $t = 1$. Moreover $H^0 = a$ is regular, fixed point-free on $A \cup B$ and $A \cup B$, as a metric space, is normal. Hence we are in position to apply the Homotopy Extension Theorem, Theorem 3.2, with respect to the pair $(A \cup B, A)$ and the homotopy $H^t|_{A\cup B} : A \cup B \longrightarrow 2^X$. Hence $H^1|_A = b$ is inessential with respect to $A \cup B$. Since B contains b this contradicts Theorem 3.1. $\square$

In presence of compactness we get rid of the uniform formulation of Theorem 3.4. I. e., the vicinity V in Theorem 3.4 can be replaced by the diagonal $\Delta(X)$. Indeed, if F^t is compact so is F^1 and it is straightforwardly shown that if a, b belong to the same component of $X \setminus (id - F^1)(A)$ and to different components of

$X \setminus A$ then also a, b belong to the same component of $X \setminus (id - VF^1V)(A)$ and to different components of $X \setminus V(A)$ for eventually all $V \in \mathcal{V}(X)$. Thus, by Theorem 3.4, for each $V \in \mathcal{V}(X)$ there exists $t_V \in I$ such that $\{a, b\} \cap (id - VF^{t_V}V)(A) \neq \emptyset$. Say $a \in (id - VF^{t_V}V)(A)$ for all $V \in \mathcal{V}(X)$. Then, by compactness of F^t, $a \in (id - F^{t_0})(A)$ for each cluster point t_0 of $(t_V)_V$.

The same is true for α-contractive homotopies f^t. If a, b belong to the same component of $X \setminus (id - f^1)(A)$ there exists an arc c from a to b which image is disjoint to $(id - f^1)(A)$. Assume to the contrary that a to b is not disjoint to $(id - Vf^1V)(A)$ for eventually all $V \in \mathcal{V}(X)$. Then for each $V \in \mathcal{V}(X)$ there exist $x_V \in (id - Vf^1V)(a_V)$ for some $x_V \in c(I)$ and $a_V \in A$. We can assume $(x_V)_V$ converges to $x \in c(I)$ and infer $a'_V \in (Vf^1V)(a'_V) + V(x)$ for some $a'_V \in A$. Since summation is uniformly continuous we infer $a''_V \in V(f^1 + x)V(a''_V)$ for some $a''_V \in A$ and, as in the proof of Proposition 2.4, the net $(a''_V)_V$ turns out to be Cauchy, since contractivity of f^1 implies contractivity of $f^1 + x$. Since A is closed, X complete and f^1 uniformly continuous we infer $x \in (id - f^1)(A)$ which contradicts $c(I) \cap (id - f^1)(A) = \emptyset$.

Now we consider classes of homotopies where regularity of (3.37) comes from the invariance of the class with respect to concatenation with arcs. The following three versions of the Sweeping Theorem are classical.

If X is a Fréchet space and $F^t : X \longrightarrow 2^X$ a convex-valued compact closed homotopy then so is H^t, the homotopy in (3.37). Hence, since $H^1 = b$ is uniformly continuous, A-regularity of the restricted concatenation (3.37) in $t = 1$ follows from Example 3.10.

The same is true if X is AE(*compact metric*) and $F^t = f^t$ a single-valued compact continuous homotopy.

Moreover we do not leave the above classes if we consider restrictions of H^t, hence we infer in addition with the above discussion of compactness

Corollary 3.3. *Let A be a closed and non-empty subset of a topological group $(X, +)$ and let $F^t : X \longrightarrow 2^X$ be a compact homotopy such that $F^0 = 0$. Suppose in addition that*

(a) X is a Fréchet space and F^t is convex-valued or

(b) X is contractible, locally contractible, metrizable, AE(compact metric) *and $F^t = f^t$ is single-valued.*

Then $\{a, b\}$ is swept by $(id - F^t)(A)$ provided that a and b belong to the same component of $X \setminus (id - F^1)(A)$ and to different components of $X \setminus A$.

We consider now α-contractive families f^t on a complete metric topological group $(X, +, d)$. We already noted that every arc $c : I \longrightarrow X$ defines a 0-contractive family by Example 3.4. It is straightforwardly shown that the sum of an α-contractive and a 0-contractive family is an α-contractive family provided that the metric d is translation-invariant. Hence α-contractiveness of a homotopy

f^t is preserved if we add an arc c as in (3.37). Moreover concatenations and restrictions of α-contractive families are α-contractive, see Remark 3.1. Thus $h^t = H^t$ in (3.37) is a α-contractive family whenever so is f^t. Finally observe that $\partial(A \cup B) \subseteq A$ where the homotopy h^t is fixed point-free. Hence, by Examples 3.5, 3.12, h^t fulfills the hypothesis on (3.37) and we infer

Corollary 3.4. *Let $(X, +, d)$ be a complete metric contractible and locally contractible topological group with translation-invariant metric d. Let $A \subseteq X$ be closed and non-empty and $f^t : X \longrightarrow X$ a bounded α-contractive homotopy such that $f^0 = 0$.*

Then $\{a, b\}$ is swept by $(id - f^t)(A)$ provided that a and b belong to the same component of $X \setminus (id - f^1)(A)$ and to different components of $X \setminus A$.

Proposition 3.5 provides a sweeping theorem which is a little bit less academic than Theorem 3.4.

Corollary 3.5. *Let $(X, +)$ be a contractible locally contractible metrizable topological group, $A \subseteq X$ closed and non-empty, $\{a, b\} \subseteq X \setminus A$ and B the component of $X \setminus A$ which contains b.*

Let $F^t : X \longrightarrow 2^X$ be a compact homotopy such that $F^0 = 0$ and $F^1 = x_0$ for some $x_0 \in X$.

Assume in addition that for each arc $c : I \longrightarrow X$, joining a and b, the restricted homotopy

$$\big(F^t + c(t)\big)\,|_{A \cup B} : A \cup B \longrightarrow 2^X \tag{3.40}$$

is $(A \cup B)$-regular in $t = 1$.

Then $\{a, b\}$ is swept by $(id - F^t)(A)$ provided that a and b belong to the same component of $x_0 + (X \setminus A)$ and to different components of $X \setminus A$.

Proof. Since F^t is compact it is sufficient to show the hypothesis on (3.37) and to apply Theorem 3.4 to infer the 'uniform version' of the Sweeping Theorem. Our aim is to show this with the help of Proposition 3.5.

Observe first that, since $F^1 = x_0$, (3.37) reads as

$$H^t = (F^{1-t} + b)(x_0 + c(t))(F^t + a) \tag{3.41}$$

and, as in the proof of Theorem 3.4, c is an arc joining a and b.

Since the constant function a is an arc we infer by assumption $(A \cup B)$-regularity of $(F^t + a)|_{A \cup B} : A \cup B \longrightarrow 2^X$ in $t = 1$. By Example 3.9 the arc $(x_0 + c(t)) : A \cup B \longrightarrow X$ is $(A \cup B)$-regular. Moreover since $F^1 + a = x_0 + a = x_0 + c(0)$ the additional continuity hypothesis of Proposition 3.5 follow by means of compactness of F^t. X enjoys the desired contractibility assumptions. We infer from Proposition 3.5, see the note at the end of its proof, $(A \cup B)$-regularity of the concatenation $(x_0 + c(t))(F^t + a)$ in $t = 1$. An analogue argumentation provides $(A \cup B)$-regularity of H^t in $t = 1$ and the hypothesis of (3.37) are fulfilled. $\square$

Apparently every arc $F^t = c'(t)$ fulfills the hypothesis of Corollary 3.5 and we infer a sweeping theorem for translations along arcs. However, there is no need for the above apparatus to show this.

4 Application to Roberts spaces

As mentioned in the introduction a long outstanding problem in fixed point theory is whether a compact convex subset K of a HTVS E is a fixed point-space: does every continuous function $f : K \longrightarrow K$ enjoy a fixed point? See the Scottish Book [Me81, Problem 54]. Due to J. Schauder [Sch30] this is known as *Schauder's Conjecture*. In 1935 A. Tychonoff [Tyc35] gave a proof for locally convex E. Starting from this a lot of effort has been made to weaken the assumption of local convexity.

From an analytical point of view local convexity of E is needed to make use of Theorem 1.4 and obtain for each $V \in \mathcal{V}(E)$ a uniform approximation $id_V : K \longrightarrow P_V$ of $id : K \longrightarrow K$: $(id, id_V)(K) \subseteq V$, where P_V is a convex polyhedron which is contained in K. The approximation $id_V f : K \longrightarrow P_V$ of f provides the selection

$$\begin{tikzcd} K \arrow[rr, "V_K f"] \arrow[dr, "id_V f"'] & & K \\ & P_V \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.1}$$

and its restriction $id_V f|_{P_V} : P_V \longrightarrow P_V$ has a fixed point x_V by the Brouwer fixed point theorem, see Theorem 1.10. By (4.1) every cluster point of $(x_V)_V$ turns out to be a fixed point of f. Since K is compact there is at least one cluster point, hence $Fix(f) \neq \emptyset$.

In 1960 V. Klee [Kle60b], [Kle60a] introduced the notion of *admissible* HTVS E which suppose the existence of approximations like id_V. Locally convex spaces are admissible. Non-locally convex admissible spaces were investigated by the schools of Aachen, Dresden and Novi Sad in the seventies. T. Riedrich [Rie63], [Rie64] showed the admissibility of the spaces $L_p[0,1]$, $0 < p < 1$ and the space $S[0,1]$ of measurable functions. J. Ishii [Ish65] generalized Riedrichs results to a variety of Orlicz function spaces. C. Krauthausen [Kra74] showed admissibility of the Hardy spaces H_p, $0 < p < 1$. In 1966 T. Riedrich [Rie66] generalized (unpublished) the notion of admissibility to subsets of HTVS (see also C. Krauthausen [Kra74]). For a survey on fixed point theory in HTVS (up to 1984) we refer to the monograph [Had84].

In 1994, investigating the fixed point property for J. W. Roberts [Rob77b], [Rob77a] examples of compact convex subsets of HTVS where the Krein-Milman theorem fails, N. T. Nhu and L. H. Tri [NT94], [Nhu96] showed the *simplicial approximation property*, see Definition 4.1 below, for Roberts spaces. Starting from this, N. T. Nhu [Nhu96] introduced the notion of *weakly admissible* convex compact subsets of HTVS which are the main motivation of our investigations.

For what follows let E be a HTVS and $\mathcal{V}(E)$ a base of circled vicinities in E.

4.1 Spaces with simplicial approximation property

We first take a look at a fundamental barycentric approximation technique which generalizes [vdBDHvdM92, Lemma 2.4] and [Oko00b, Lemma 1].

Lemma 4.1. *Let K be a non-empty compact convex subset of E, Q a polyhedron and $\hat{Q} = \cup_{i=1}^{n} \hat{Q}_i$ a pairwise disjoint union of compacta $\hat{Q}_i \subseteq Q$. Let $F : Q \longrightarrow 2^K$ be a closed map with non-empty convex values.*

Then for every $V \in \mathcal{V}(E)$ and every selection $\hat{f} : \hat{Q} \longrightarrow K$ of F which is constant on every $\hat{Q}_i$ there exists a selection

$$\begin{tikzcd} Q \arrow[rr, "V_K F"] \arrow[dr, "f"'] & & K \\ & P \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.2}$$

which extends $\hat{f}$. Moreover P is a convex polyhedron.

Proof. Since $F : Q \longrightarrow 2^K$ is uniformly upper semicontinuous there exists a metric ρ on Q, which induces the Euclidean topology on Q, and an $\varepsilon > 0$ such that

$$\mathcal{G}(V_K F U_\varepsilon) \subseteq \mathcal{G}(V_K V_K F). \tag{4.3}$$

Hence it is sufficient to show (4.2) for $V_K F U_\varepsilon$ instead of $V_K F$.

Let d be the (covering) dimension of Q and $V' \in \mathcal{V}(E)$ such that $V' \underbrace{\cdots\cdots}_{2d\text{times}} V' \subseteq V$. Choose $0 < \varepsilon' \leq \varepsilon$ such that

$$\mathcal{G}(F U_{\varepsilon'}) \subseteq \mathcal{G}(V'_K F). \tag{4.4}$$

Since the $\hat{Q}_i$ are pairwise disjoint and compact there exists $0 < \varepsilon'' \leq \varepsilon'$ such that their neighborhoods $U_{\varepsilon''}(\hat{Q}_i)$ are pairwise disjoint, too. Consider the open covering $\mathfrak{Q}$ of Q given by $\{U_{\varepsilon''}(q) \, ; \, q \in \hat{Q}\} \cup \{U_{\varepsilon''}(q) \setminus \hat{Q} \, ; \, q \in Q \setminus \hat{Q}\}$. $\mathfrak{Q}$ admits a finite open point-star-refinement $\mathfrak{Q}'$ with at most d-dimensional nerve, see e. g. [vM89, 4.3.5. Theorem].

For each $Q' \in \mathfrak{Q}'$ choose $q' \in Q'$ such that $q' \in \hat{Q}_i$ if $Q' \cap \hat{Q}_i \neq \emptyset$ for some i. This is well defined by the choice of ε'' and since $\mathfrak{Q}'$ is a refinement of $\mathfrak{Q}$. Let α be the barycentric function $\alpha : Q \longrightarrow |\mathfrak{Q}'|$ with vertices q', i. e.

$$\alpha(q) := \left(\sum_{Q' \in \mathfrak{Q}'} dist(q, Q \setminus Q') \right)^{-1} \sum_{Q' \in \mathfrak{Q}'} dist(q, Q \setminus Q') q', \quad q \in Q,$$

where $dist(q, \tilde{Q}) := \inf\{\rho(q, \tilde{q}) \, ; \, \tilde{q} \in \tilde{Q}\}$. For each vertex q' of $\mathfrak{Q}'$ choose $k' \in (V'_K F)(q')$ such that $k' = \hat{f}(q')$ if $q' \in \hat{Q}$. This is possible since $\hat{f}$ is constant on each $\hat{Q}_i$. Let $\beta : |\mathfrak{Q}'| \longrightarrow K$ be the affine extension of the function defined by $q' \mapsto k'$ on the vertices q' of $\mathfrak{Q}'$. The composition $f := \beta\alpha$ is continuous and maps to the convex polyhedron $P := conv\,(\{k'\,;\, Q' \in \mathfrak{Q}'\})$ which is contained in K. Here *conv* denotes the convex hull. Furthermore $\mathcal{G}(\hat{f}) \subseteq \mathcal{G}(f)$ by the choice of the coverings and since $\hat{f}$ is constant on each $\hat{Q}_i$.

It remains to show the factorization (4.2). Let $q \in Q$. Then there exist $\alpha_{Q'} \in I$ such that $\sum_{Q' \in \mathfrak{Q}'} \alpha_{Q'} = 1$ and

$$f(q) = \sum_{\substack{Q' \in \mathfrak{Q}' \\ q \in Q'}} \alpha_{Q'} k' \in \sum_{\substack{Q' \in \mathfrak{Q}' \\ q \in Q'}} \alpha_{Q'} (V'_K F)(q') \subseteq \sum_{\substack{Q' \in \mathfrak{Q}' \\ q \in Q'}} \alpha_{Q'} (V'_K F)(Q'). \tag{4.5}$$

Since $\mathfrak{Q}'$ is a point-star-refinement of $\mathfrak{Q}$ there exists $\tilde{q} \in Q$ such that $(q, \tilde{q}) \in U_{\varepsilon''}$ and $Q' \subseteq U_{\varepsilon''}(\tilde{q})$ for all non-vanishing $\alpha_{Q'}$ in (4.5). Hence we infer from (4.5)

$$f(q) \in \sum_{\substack{Q' \in \mathfrak{Q}' \\ Q' \subseteq U_{\varepsilon''}(\tilde{q})}} \alpha_{Q'} (V'_K F)(U_{\varepsilon''}(\tilde{q})) \subseteq \sum_{\substack{Q' \in \mathfrak{Q}' \\ Q' \subseteq U_{\varepsilon'}(\tilde{q})}} \alpha_{Q'} (V'_K V'_K F)(\tilde{q}) \tag{4.6}$$

where the latter inclusion comes from (4.4) and $\varepsilon'' \leq \varepsilon'$. Since the nerve of $\mathfrak{Q}'$ is at most d-dimensional there are at most d non-vanishing summands in (4.6) and since all $\alpha_{Q'}$ are elements of I, V' is circled and $F(\tilde{q})$ convex we infer from (4.6)

$$f(q) \in \sum_{j=1}^{d} (V'_K V'_K F)(\tilde{q}).$$

Moreover $(\tilde{q}, q) \in U_\varepsilon$ since $0 < \varepsilon' \leq \varepsilon$. By (4.3) and the choice of V' we have shown (4.2). $\square$

The above Lemma differs from classical barycentric approximation only in the fact that, due to our concept of approximation, (2.1) we have to take care of preciseness of the selection (4.2).

For the same reason we reformulate the notion of *simplicial approximation property* which is (for metric spaces) from the F-space sampler [KPR84, Chapter 9, 4.].

Definition 4.1. A non-empty convex compact subset K of a HTVS E is said to have the *simplicial approximation property* if for every $V \in \mathcal{V}(E)$ there exists a convex polyhedron $P' \subseteq K$ such that for every polyhedron $P \subseteq K$ there exists a selection

$$\begin{tikzcd} P \arrow[rr, "V_K|_P"] \arrow[dr, "\psi"'] & & K \\ & P' \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.7}$$

Short hand for we call a compact convex subset of a HTVS which enjoys the simplicial approximation property a *space with simplicial approximation property.*

To customize (4.7) suitable to show preciseness observe first that we can confine ourselfs to such polyhedra P which contain some fixed finite subset $X' \in \mathcal{A}(K)$. Then it is possible to claim in (4.7) $\psi = id$ on the X'. We obtain the following equivalent formulation of the simplicial approximation property. Its proof relies on the fact that the polyhedron P' enjoys a locally convex topology. It runs in the same manner as the proof of our (customized) Schauder projection, see Theorem 1.4.

Lemma 4.2. *A space K is a space with simplicial approximation property iff for every $X' \in \mathcal{A}(K)$ and every $V \in \mathcal{V}(E)$ there exists a convex polyhedron $P' \subseteq K$ such that for every polyhedron $P \subseteq K$ which contains X' there exists a selection*

$$\begin{tikzcd} P \arrow[rr, "V_K|_P"] \arrow[dr, "\psi"'] & & K \\ & P' \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.8}$$

which extends the identity on X'.

It turns out that closed convex-valued maps and homotopies on spaces K with simplicial approximation property are regular. More precisely is

Proposition 4.1. *Let K be a space with simplicial approximation property and $A \subseteq K$ non-empty and closed. Let $F : A \longrightarrow 2^K$ and $F^t : A \longrightarrow 2^K$ be a closed and convex-valued map and homotopy, respectively. Then F and F^t are regular.*

Moreover F^t is A-regular in each $t' \in I$ where $F^{t'}$ is a constant.

Proof. First we show regularity. By Proposition 3.1 is is sufficient to consider the homotopy F^t. Since K is compact by Corollary 2.6 for homotopies it is sufficient to consider the case $A = K$, moreover the subnet condition is superfluous. Hence, to prove regularity, it remains to show approximability of $F^t : K \longrightarrow 2^K$.

Fix $(X', V_K) = \gamma \in \Gamma(K)$ and choose $V' \in \mathcal{V}(E)$ such that $V'_K V'_K \subseteq V_K$. Lemma 4.2 provides a convex polyhedron P' with $X' \subseteq P' \subseteq K$ such that for every polyhedron P with $X' \subseteq P \subseteq K$ there exists a selection

$$\begin{tikzcd} P \arrow[rr, "V'_K|_P"] \arrow[dr, "\psi_P"'] & & K \\ & P' \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.9}$$

which extends the identity on X'.

For each $x' \in X'$ let $J(x') := \{t \in I\,;\, x' \in F^t(x')\}$. Then $\cup_{x' \in X'}(J(x') \times \{x'\})$ is a pairwise disjoint union of compacta in the polyhedron $I \times P'$. By Lemma

4.1, applied to the restriction $\tilde{F}|_{I\times P'}$ of the map $\tilde{F}$, which is associated to the homotopy F^t, we obtain a selection

$$\begin{tikzcd} I \times P' \arrow[rr, "V'_K \tilde{F}|_{I\times P'}"] \arrow[dr, "\alpha"'] & & K \\ & P \arrow[ur, hook, "\iota"'] & \end{tikzcd} \tag{4.10}$$

through a polyhedron P such that

$$\alpha \text{ extends the (constant) functions } (J(x') \times \{x'\}) \mapsto x', \quad x' \in X'. \tag{4.11}$$

By the Tietze extension theorem, see Theorem 1.2, α can be extended to $\hat{\alpha} : I \times K \longrightarrow \hat{P}$.

Fix now the polyhedron P in (4.9) to be the same as that in (4.10) and choose a space $\hat{P}$, being homeomorphic to a suitable cube I^n, which contains P. Extend ψ_P by the Dugundji extension theorem, see Theorem 1.3, to $\hat{\psi} : \hat{P} \longrightarrow P'$.

We obtain the diagram

$$\begin{tikzcd} I \times K \arrow[rr, dotted, "V'_K V'_K \tilde{F}"] \arrow[dr, "\hat{\alpha}"] & & K \\ I \times P' \arrow[u, hook] \arrow[r, "\hat{\alpha}"] \arrow[dr, "\alpha"] \arrow[d, "V'_K \tilde{F}"'] & \hat{P} \arrow[r, "\hat{\psi}_P"] \arrow[ur, "\hat{\psi}_P"] \arrow[ul, phantom, "(i)"] \arrow[ur, phantom, "(ii)"'] \arrow[dl, phantom, "(iii)"] \arrow[dr, phantom, "(iv)"] & P' \arrow[u, hook] \arrow[d, hook'] \\ K & P \arrow[l, hook', "\iota"'] \arrow[u, hook] \arrow[ur, "\psi_P"] \arrow[r, "V'_K|_P"] \arrow[ul, phantom, "(v)"] \arrow[ur, phantom, "(vi)"'] & K \end{tikzcd} \tag{4.12}$$

with selections/factorizations (i) to (vi).

Our aim is to put the γ into the 'open' upper triangle of (4.12). Then

$$\begin{tikzcd} K \arrow[rr, "F^t"] \arrow[dr, "\hat{\alpha}^t"'] & \gamma & K \\ & \hat{P} \arrow[ur, hook, "\hat{\psi}_P"'] & \end{tikzcd} \tag{4.13}$$

where $\hat{\alpha}^t$ is the homotopy induced by $\hat{\alpha}$.

In fact this is possible. Being homeomorphic to a cube, $\hat{P}$ is a convex polyhedron and apparently $\hat{\alpha}^t$ and $\hat{\psi}_P$ are continuous. To see V_K-proximity let $x = (\hat{\psi}_P \hat{\alpha}^t)(x)$. Then $x \in P'$ by factorization (ii), hence $x = (\psi_P \alpha)(t, x)$ by factorizations (iii) and (iv) and $x \in (V'_K V'_K \tilde{F})(t, x)$ by selections (v) and (vi). Since $V'_K V'_K \subseteq V_K$ we have shown V_K-proximity.

To see preciseness let $x' \in X' \cap Fix((F^t))$, i. e. $x' \in F^{t'}(x')$ for some $t' \in I$. Then $(t', x') \in J(x') \times \{x'\}$ and $\alpha(t', x') = x'$ by (4.11). Moreover, since $X' \subseteq P$ and ψ_P extends the identity on X' so does $\hat{\psi}_P$ by factorization (iv). Hence $\hat{\psi}_P(x') = x'$ and $x' = (\hat{\psi}_P \hat{\alpha}^{t'})(x')$ which is preciseness.

We have shown regularity.

Let F^t be a constant for some $t \in I$, say $F^{t'} = x'$. Since there is only a minor difference to the above proof of regularity we give only a sketch of the proof of A-regularity.

Again we can confine ourselfs to the case $A = K$ and have to show K-approximability.

We modify the above proof as follows: 1st. Add x' to X' to obtain $\psi_P^{x'}$ instead of ψ_P in (4.9). 2nd. Choose a selection $\alpha^{x'}$ in (4.10) such that, in addition to (4.11), α^{x_0} extends the constant function $(\{t'\} \times P') \mapsto x'$, too. This is possible without mismatch to (4.11) since $Fix(F^{t'}) \subseteq \{x'\}$. 3rd. Before making use of the Tietze theorem extend $\alpha^{x'}$ to $\alpha' : (I \times P') \cup (\{t'\} \times K) \longrightarrow P$ given by $\alpha'(t, x) = \alpha^{x'}(t, x)$ if $t \neq t'$ and $\alpha'(t', x) = x'$, $x \in K$.

From this modification which is without mismatch to the above proofs of V_K-proximity and preciseness with respect to X' we infer $\hat{\psi}_P \hat{\alpha}_P^{t'} = x'$, hence K-approximability in t'.

Uniform continuity of the constant function $F^{t'}$ is obvious since K is compact. □

The first immediate consequence of Proposition 4.1 is the Kakutani fixed point theorem for spaces with simplicial approximation property. More precisely Theorem 2.2 provides

Theorem 4.1. *Let K be a space with simplicial approximation property and $F : K \longrightarrow 2^K$ closed and convex-valued. Then F has a fixed point $x_0 \in F(x_0)$.*

Since every convex subset of a HTVS is contractible and locally contractible we infer from Corollary 3.2 a Nonlinear Alternative.

Theorem 4.2. *Let K be a metrizable space with simplicial approximation property, $A \subseteq K$ closed and $x_0 \in \mathring{A}$. Let $F : A \longrightarrow 2^K$ be closed and convex-valued. Then at least one of the following statements holds:*

(i) F has a fixed point.

(ii) There exists $0 < t < 1$ and $x \in \partial A$ such that $x \in tx_0 + (1-t)F(x)$.

Proof. Apply Corollary 3.2 to the closed convex-valued homotopy $F^t : A \longrightarrow 2^K$ given by $F^t(x) := tx_0 + (1-t)F(x)$, $x \in A$. Since $F^1 = x_0$ this homotopy is A-regular in $t = 1$ by Proposition 4.1. □

Remark 4.1. We remark that Theorem 4.2 is far from being new. Indeed, it can be derived by means of classical theorems of Topological Transversality, see e. g. [Gra01] or [Jer82], since that by Theorem 4.1 any space with simplicial approximation property is a fixed point space for closed convex-valued maps. C. f. our discussion in Remark 3.4, too.

4.2 Admissibility

N. T. Nhu [Nhu96] defines weak admissibility for metrizable HTVS E and convex compact subsets K of E. We generalize his definition to obtain a more easy classification into the historical context.

Definition 4.2. A non-empty compact convex subset K of a HTVS E is said to be *weakly admissible* if for every $V \in \mathcal{V}(K)$ there exists $n \in \mathbb{N}$, $K_i \subseteq K$ compact convex and continuous functions $I_i : K_i \longrightarrow K$, $i = 1, \ldots, n$ such that

(i) $span\, I_i(K_i)$ is finite dimensional, $\quad i = 1, \ldots, n$,

(ii) $conv\,(\cup_{i=1}^n K_i) = K$,

(iii) $\sum_{i=1}^n (I_i(x_i) - x_i) \in V(0), \quad (x_1, \ldots, x_n) \in \prod_{i=1}^n K_i$.

Here *span* denotes the linear hull.

Again, short hand for, we use the term *weakly admissible space* for compact convex subsets of HTVS which are weakly admissible.

Weak admissibility is closely related to the notion of *admissible spaces* of V. Klee [Kle60b], [Kle60a] and of admissible subsets of HTVS in the sense of T. Riedrich [Rie66] and C. Krauthausen [Kra74] where in both cases n is claimed to be 1. More precisely is

Definition 4.3. A non-empty convex subset K of a HTVS E is said to be *admissible* if for every compact $K' \subseteq K$ and $V \in \mathcal{V}(K)$ there exists a continuous selection $I' : K' \longrightarrow K$ of $V|_{K'}$ such that $span\, I'(K')$ is finite dimensional.

We will use the term *admissible space* for convex subsets of HTVS which are admissible.

Apparently every compact convex admissible space is a weakly admissible space.

Let K be a compact admissible subset of a HTVS E and $x_0 \in E \setminus K$. Up to the best knowledge of the author it is unknown whether $conv\,(K \cup \{x_0\})$ is admissible, too. In contrast to this for weakly admissible $K_i \subseteq E$, $i = 1, \ldots, n$ we infer apparently weak admissibility of $conv\,(\cup_{i=1,\ldots,n} K_i)$.

For metrizable HTVS E N. T. Nhu [Nhu96] showed

Lemma 4.3. *Every compact weakly admissible space is a space with simplicial approximation property.*

The author [Oko00b] generalized this to arbitrary HTVS and shows the Kakutani fixed point theorem and the Nonlinear Alternative for closed convex-valued maps/homotopies in compact weakly admissible spaces K. Observe that our approach by means of regular maps/homotopies and Theorems 4.1, 4.2 provides these results only for metrizable K.

4.3 Roberts spaces

J. W. Roberts [Rob77b], [Rob77a] gave the, up to the best knowledge of the author only one, example of a compact convex subset K of a HTVS E where the Krein-Milman theorem fails, i. e. K is not the closed convex hull of the extreme points of K. His example is constructive in character and makes use of so called *needle points* which can be motivated as follows.

Literature to this topic mainly deals with F-spaces E, i. e. metrizable HTVS. However, at least most of the following definitions make sense for non-metrizable E, too. Hence we keep on using the vicinities $\mathcal{V}(E)$ instead of a pure metric formulation.

Let E be a HTVS, E' its (topological) dual and E'' its bidual. If E' is *total*, i. e. E' separates the points of E, each compact convex subset K of E embeds affinely in E'' by restriction of the natural embedding $\iota_E : E \hookrightarrow E''$. Endow E'' with the topology τ of pointwise convergence. Then $\iota_E|_K : K \longrightarrow \iota_E(K)$ is in addition a homeomorphism. Hence, since (E'', τ) is locally convex, in every HTVS having total dual the Krein-Milman theorem holds.

We have the following characterization of a trivial dual: $E' = \{0\}$ iff for every $x \in E \setminus \{0\}$ and every $V \in \mathcal{V}(E)$ there exists $\{x_1, \ldots, x_n\} \subseteq E$ such that

$$(i) \quad \{x_1, \ldots, x_n\} \subseteq V(0) \qquad \text{and} \qquad (ii) \quad x \in V(conv\,\{x_1, \ldots, x_n\}). \tag{4.14}$$

However, N. J. Kalton [Kal80] showed that there are convex compact subsets of certain Orlicz function spaces L_ϕ with trivial dual where the Krein-Milman theorem holds. Kaltons argumentation relies on the notions of *approachable points* and *locally convex subsets* of HTVS.

The notion of approachable points in HTVS is due to N. J. Kalton [Kal80]. N. J. Kalton and N. T. Peck [KP80] defined a similar term for convex subsets K of HTVS E which coincides with Kaltons definition for locally bounded F-spaces. However, to illustrate Kaltons result [Kal80] we make use of the following definition which is equivalent to Kaltons definition for $K = E$.

Let E be a HTVS and K a closed convex subset of E which contains the origin 0. A point $x \in K$ is said to be an *approachable point of* K if there exists a bounded subset B of K such that for every $V \in \mathcal{V}(E)$ there exists $\{x_1, \ldots, x_n\} \subseteq E$ such that, in addition to (i) and (ii) of (4.14),

$$(iii) \quad conv\,\{x_1, \ldots, x_n\} \subseteq B. \tag{4.15}$$

If K is compact we apparently can choose $B = K$ in (4.15).

Locally convex subsets are due to C. Krauthausen [Kra76].

A convex subset K of a HTVS is said to be *locally convex* if every $x \in K$ enjoys a base of convex neighborhoods in K.

N. J. Kalton [Kal80, Theorem 1] showed for a metrizable and complete HTVS E that if for a compact convex subset K of E, containing the origin 0, 0 is the only approachable point of K then K is locally convex. He constructed an Orlicz space L_ϕ with trivial dual where 0 is the only approachable point (in L_ϕ). See Remark 4.2 for some basics regarding Orlicz spaces.

The following characterization of locally convex compact convex subsets of HTVS is from [Web92]. See also the references therein.

Proposition 4.2. *Let K be a compact convex subset of a HTVS E. Then the following statements are equivalent:*

(i) K is locally convex,

(ii) K is affinely embeddable into a locally convex space,

(iii) the continuous affine functionals on K separate the points of K.

Hence, even though there exists no non-trivial continuous linear functional there can be enough affine ones on each compact convex subset K of E to ensure the Krein-Milman theorem.

The crucial idea how to construct a compact convex subset K of a HTVS E which is not embeddable into a locally convex space is to claim that B in (4.15) is approximately the segment from 0 to x.

More precisely for a closed convex subset K of a HTVS E a point $x \in K$ is said to be a *needlepoint of* K if for every $V \in \mathcal{V}(E)$ there exists $\{x_1, \ldots, x_n\} \subseteq K$ such that, in addition to (i) and (ii) of (4.14),

$$(iii') \quad conv\,\{x_1, \ldots, x_n\} \subseteq V([0, x]). \tag{4.16}$$

Observe that (iii') is in fact a generalization of (iii) if we are concerned with a compact K. A HTVS E is said to be a *needlepoint space* if all $x \in E \setminus \{0\}$ are needlepoints of E.

Besides Roberts original papers detailed describtions of what follows can be found in [KP80], [Rol85, 5.6] or in the F-space sampler [KPR84, Chapter 9]. We follow [Rol85, 5.6].

Proposition 4.3. *In every metrizable complete needlepoint space E a compact convex subset K_R without extreme points can be constructed.*

Proof. Choose a monotonous F-norm $\|\cdot\|$ on E which induces a metric in E compatible with the topology of E. For any $\varepsilon > 0$ $V_\varepsilon(0) = \{x \in E\,;\, \|x\| < \varepsilon\}$ denotes the open ε-ball around the origin. Following literature let us agree to call the set $\{x_1, \ldots, x_n\}$ of (4.14), (4.16), where $V = V_\varepsilon$, an *ε-needleset for x*. Choose a sequence $(\varepsilon_n)_n$ of positive reals which are summable, $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, and define inductively as follows.

Choose an initial (needle)point $x^0 \in E \setminus \{0\}$ and let $X^0 := \{x^0\}$. Let $X^1 = \{x_1^0, \ldots, x_{n_1}^0\}$ be an ε_0-needleset for x^0. For each $x_i^0 \in X^1$ choose an (ε_1/n_1)-needleset X_i^1 and let $X^2 = \{x_1^1, \ldots, x_{n_2}^1\}$ be their union. In general:

$$\begin{aligned}&\text{for each } x_i^{n-1} \in X^n\\ &\text{choose an } (\varepsilon_n/n_n)\text{-needleset for } X_i^n\\ &\text{and let } X^{n+1} = \{x_1^n, \ldots, x_{n_{n+1}}^n\} \text{ be their union.}\end{aligned} \tag{4.17}$$

From (4.17), (4.16) we infer

$$conv\ \big(X^{n+1} \cup \{0\}\big) \subseteq V_{\varepsilon_n}\big(conv\ (X^n \cup \{0\})\big), \quad n = 0, 1, \ldots \tag{4.18}$$

Let

$$K := \overline{conv \bigcup_{n=0}^{\infty} (X^n \cup \{0\})}. \tag{4.19}$$

By the invariant (4.18) and since $(\varepsilon_n)_n$ is summable for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\sum_{n=n_\varepsilon}^{\infty} \varepsilon_n \leq \varepsilon$ and

$$K \subseteq \overline{V}_\varepsilon\big(conv\ (X^{n_\varepsilon} \cup \{0\})\big). \tag{4.20}$$

Hence K is precompact and since E is assumed to be complete also compact. Moreover, only the accumulation points of $\cup_n X^n$ can be extreme points of K. Hence, since $\varepsilon_n \longrightarrow 0$, only 0 can be an extreme point of K. Now let

$$K_R := conv\,(K \cup (-K)) \tag{4.21}$$

to get rid of this last candidate. Since K_R is the absolute convex hull of K it is a convex compact subset of E since so is K. $\square$

In literature a frequently-used term for the space K_R, constructed as in (4.17) to (4.21), is *Roberts space*. In what follows we make use of this term, too.

It remains to show the existence of a needlepoint space E. In fact, this is one of Roberts major results. However, to prove that (i) to $(iii)'$ hold for some $x \in E \setminus \{0\}$ is highly technical and out of place here. So far note only that if ϕ is an Orlicz function then

$$\liminf_{x\to\infty} \frac{\phi(x)}{x} = 0 \quad \text{and} \quad \limsup_{x\to\infty} \frac{\phi(x)}{x} < \infty$$

is a sufficient condition for the assigned Orlicz space L_ϕ to be a needlepoint space. Particularly for every $0 < p < 1$ the space L_p of all to the power p integrable functions is a needlepoint space.

Further examples of needlepoint spaces are obtained from quotient spaces. [KPR84, Theorem 9.7] states that for every $0 < p < 1$ the quotient $H_p/S_\mu H_p$, where H_p is the Hardy space and S_μ a singular inner function in H_p, is a needlepoint space. See e. g. [Dur70] for basics regarding Hardy spaces.

We stop our considerations of the Roberts spaces and bridge to weakly admissible spaces.

Lemma 4.4. *Every Roberts space K_R is weakly admissible.*

Proof. Let E be the needlepoint space where the Roberts space K_R is constructed. We have to show the existence of approximations $I_i' : K_i' \longrightarrow K_R$ as in Definition 4.2 where $V = V_\varepsilon$ for some given $\varepsilon > 0$. We make use of the notation in the proof of Proposition 4.3, in particular those of (4.17) to (4.21).

By (4.21) it is sufficient to show weak admissibility of K given by (4.19). Fix $\varepsilon > 0$ and choose n_ε as in (4.20). Consider X^{n_ε}. By (4.17) for each $x_i^{n_\varepsilon - 1} \in X^{n_\varepsilon}$ we chose $(\varepsilon_{n_\varepsilon}/n_{n_\varepsilon})$-needlesets $X_i^{n_\varepsilon}$. Fix one of them, say $X_{i_0}^{n_\varepsilon}$. We carried on with the choice of needlesets for all elements x_i of $X_{i_0}^{n_\varepsilon}$. Proceeding in this manner we obtain a subset $\tilde{X}_{i_0}$ of $\cup_n X^n$ assigned to the needleset $X_{i_0}^{n_\varepsilon}$.

Use the same procedure for each of the $X_i^{n_\varepsilon}$ to obtain a partition $\cup_n X^n = \tilde{X}_1 \cup \ldots \cup \tilde{X}_{n_\varepsilon}$ and the representation

$$K = \overline{conv \bigcup_{n=0}^{\infty} (X^n \cup \{0\})} = conv \left(\bigcup_{i=1}^{n_\varepsilon} \overline{conv\,(\tilde{X}_i \cup \{0\})} \right). \tag{4.22}$$

From property (4.16) of the needlesets we infer likewise (4.20)

$$conv \left(\tilde{X}_i \cup \{0\} \right) \subseteq \overline{V}\,(\varepsilon/n_\varepsilon) \left([0, x_i^{n_\varepsilon - 1}] \right), \quad i = 1, \ldots, n_\varepsilon. \tag{4.23}$$

Define

$$K_i := \overline{conv \left(\tilde{X}_i \cup \{0\} \right)}, \quad i = 1, \ldots, n_\varepsilon$$

which are convex compact subsets of K and hold (ii) of Definition 4.2 by (4.22).

A quantitative analyse of the proof of Dugundji's extension theorem, see Theorem 1.3, provides retractions $r_i : E \longrightarrow [0, x_i^{n_\varepsilon - 1}]$ such that

$$\|x - r_i(x)\| \leq 4 \inf_{y \in [0, x_i^{n_\varepsilon - 1}]} \|x - y\|, \quad x \in E \tag{4.24}$$

for $i = 1, \ldots, n_\varepsilon$. (A detailed construction of the r_i gives [NT94, Lemma 1].) Define $I_i := r_i|_{K_i}$, $i = 1, \ldots, n_\varepsilon$. These I_i are continuous functions mapping to $span\{x_i^{n_\varepsilon - 1}\}$, $i = 1, \ldots, n_\varepsilon$, respectively, hence (i) of Definition 4.2 holds.

It remains to show (iii) of Definition 4.2. Let $(y_1, \ldots, y_{n_\varepsilon}) \in \Pi_{i=1}^{n_\varepsilon} K_i$. Then

$$\sum_{i=1}^{n_\varepsilon} \|I_i(y_i) - y_i\| \leq 4 \sum_{i=1}^{n_\varepsilon} \inf_{y \in [0, x_i^{n_\varepsilon - 1}]} \|y_i - y\| \tag{4.25}$$

by (4.24). The right hand side of (4.25) is in view of (4.23) bounded by $4\sum_{i=1}^{n_\varepsilon}$ $(\varepsilon_{n_\varepsilon}/n_\varepsilon) = 4\varepsilon$.

Since ε was arbitrary we have shown weak admissibility of K, hence of the Roberts space K_R. □

In view of Lemma 4.3 and Theorems 4.1 and 4.2 the Kakutani fixed point theorem and the Nonlinear Alternative hold for every Roberts space and closed convex-valued homotopies.

Remark 4.2. For the sake of completeness we list some basics regarding Orlicz spaces. More background can e. g. be found in in the F-space sampler [KPR84] or [Rol85]. See moreover the monographs [Pal70] and [Mac72].

An *Orlicz function* ϕ is an non-decreasing function $\phi : [0,\infty) \to [0,\infty)$ which is continuous in 0, holds $\phi(x) = 0$ iff $x = 0$ and the Δ_2-condition, i. e. for some constant $\beta \geq 1$ we have $\phi(2x) \leq \beta\phi(x)$, $x \in [0,\infty)$.

Let (Ω, Σ, μ) be a finite measure space. The linear space

$$L_\phi = \left\{ f : \Omega \longrightarrow \mathbb{R};\ f \text{ measurable, } \|f\|_\phi := \int \phi(|f|)d\mu < \infty \right\}$$

is said to be an *Orlicz space*. L_ϕ turns out to be a complete metrizable HTVS with monotonous F-norm $\|\cdot\|_\phi$. Particularly for $\phi(x) = x^p$, $0 < p < 1$ and $\phi(x) = x/(1+x)$ we obtain L_p, $0 < p < 1$ and S, the spaces of to the power p integrable functions and measurable functions, respectively.

By S. Rolewicz [Rol59] locally bounded Orlicz spaces are completely characterized in terms of the Orlicz function ϕ. See also [KPR84, Theorem 2.13]. L_ϕ is locally bounded iff

$$\lim_{\lambda\to 0+} \lim_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)} = 0. \tag{4.26}$$

Observe that a sufficient condition for (4.26) is given if we twist the Δ_2-condition. In fact, if there exists $\gamma, \alpha > 1$ such that $\alpha\phi(x) \leq \phi(2x)$, $x \geq \gamma$ (4.26) holds. Thus the spaces L_p, $0 < p < 1$ are locally bounded and S fails to be.

For every non-atomic measure μ the quality of L_ϕ to have a trivial dual $(L_\phi)'$ is completely characterized in terms of the Orlicz function ϕ, too. By S. Rolewicz [Rol59] $(L_\phi)' = \{0\}$ iff

$$\liminf_{x\to\infty} \frac{\phi(x)}{x} = 0. \tag{4.27}$$

Thus $(L_p)' = \{0\}$, $0 < p < 1$ and $S' = \{0\}$. See also [KPR84, Theorem 2.12] or S. Rolewicz [Rol85, 4.4.2].

5 Further examples

The examples of regular maps and homotopies in Chapters 2 and 3 are classical. This chapter compares our approximation concept with more recent developements in fixed point theory.

5.1 Generalizations of contractiveness

Let (X, d) be a metric space. A function $f : X \longrightarrow X$ is said to be *non-expansive* if

$$d(f(x), f(y)) \leq d(x, y), \quad x, y \in X. \tag{5.1}$$

There are several fixed point theorems for non-expansive functions if X enjoys a additional geometric quality. By A. Kirk [Kir65], see also D. Goehde [Goe65] and F. E. Browder [Bro65], every non-empty closed bounded convex subset X of a uniformly convex Banach space has the fixed point property for non-expansive functions. However, in general the set of fixed points need not be compact, hence, by Theorem 2.6, non-expansive functions cannot be regular in general.

Therefore we do not go further in this direction and focus on (5.1).

As a first attempt we replace $\leq$ in (5.1) by $<$ for $x \neq y$ to obtain a more restrictive condition on $Fix(f)$. In fact, in this situation $Fix(f)$ can be at most a singleton and existence of a fixed point can e. g. be ensured by a geometric quality of X as above. Hence, by Theorem 2.4, approximability turns out to be redundant and Example 2.7 was the first illustration of this situation. However, in absence of compactness such an f fails to be regular in general.

Example 5.1. Let X be the unit ball in the sequence space l^2 of square summable real-valued sequences $(x_n)_n$ with norm $\|(x_n)_n\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$. Let $f : X \longrightarrow X$ be given by

$$f((x_n)_n) := \left(x_n - \frac{x_n}{n}\right)_n, \quad (x_n)_n \in X.$$

Then $\|f(x) - f(y)\| < \|x - y\|$ for all $x \neq y$ but f fails to be regular.

Proof. $\|f(x) - f(y)\| < \|x - y\|$, $x \neq y$ is straightforwardly shown and we focus on regularity.

For any $m \in \mathbb{N}$ and any $t \in I$ let $(t\delta_{m,n})_n = (0, \ldots, 0, t_{m\text{th}}, 0, \ldots)$ and define $\psi_m : I \longrightarrow X$ by $\psi_m(t) := (t\delta_{m,n})_n$. We claim that for each $(X', V) = \gamma \in \Gamma(X)$

there exists $m \in \mathbb{N}$ such that

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\|\cdot\|"'] & \gamma & X \\ & I \arrow[ur, "\psi_m"'] & \end{tikzcd} \tag{5.2}$$

In fact, I is a convex polyhedron and $\|\cdot\|$ and ψ_m are continuous. We can assume $V = V_\varepsilon$ for some $\varepsilon > 0$. There exists $m \in \mathbb{N}$ such that $1/m < \varepsilon$ and hence $\|(t\delta_{m,n})_n - f((t\delta_{m,n})_n)\| = t/m < \varepsilon$ for all $t \in I$. Thus $\{(t\delta_{m,n})_n \,;\, t \in I\} \subseteq Fix(V_\varepsilon f V_\varepsilon)$ which is V_ε-proximity of (5.2). Since 0 is the unique fixed point of f and $0 = \|0\|(\delta_{m,n})_n = (\psi_m\|\cdot\|)(0)$ we infer preciseness of (5.2), too. Hence $(\psi_m, \|\cdot\|) \in (f,\gamma)$ for sufficiently large m. But since $(\delta_{m,n})_n \in Fix(\psi_m\|\cdot\|)_n$ for each $m \in \mathbb{N}$ we obtain a contradiction to the subnet condition since there is no strongly convergent subnet of $((\delta_{m,n})_n)_m$ in X. □

Remark 5.1. Of course the above example lives on the fact that we chose the wrong topology for X. We obtain regularity of f if the topology of X is induced by the weak topology τ_w of l^2. In fact, f is $(\tau_w \times \tau_w)$-closed and X is τ_w-compact, hence regularity follows from Theorems 2.3 and 2.4.

Observe that it is moreover possible to define a metric d on X such that (X,d) is a complete metric space, f is contractive with respect to d and hence f is regular by Proposition 2.4. Indeed, 0 being the unique fixed point of each f^n, $n \in \mathbb{N}$ is, by a famous result of C. Bessaga [Bes59], a necessary and sufficient condition for the existence of a metric d on X such that (X,d) is a complete metric space and f contractive with respect to d.

The following definition is due to J. Dugundji and A. Granas [DG78]. See [Fri96], too.

Definition 5.1. Let (X,d) be a metric space and $A \subseteq X$ non-empty and closed. A function $\eta : A \times A \longrightarrow [0,\infty)$ is said to be *compactly positive* if

$$\theta(a,b) := \inf\{\eta(x,y)\,;\, a \le d(x,y) \le b\} > 0 \tag{5.3}$$

for each $0 < a \le b$.

A function $f : A \longrightarrow X$ is said to be *weakly contractive* if there exists a compactly positive function η such that

$$d(f(x),f(y)) \le d(x,y) - \eta(x,y), \quad x,y \in A. \tag{5.4}$$

Since for each $0 \le \alpha < 1$ the function $\eta(x,y) = (1-\alpha)d(x,y)$, $x,y \in A$ is compactly positive every contractive function turns out to be weakly contractive. If switch now to another metric, say d', it may happen that contractiveness disappears and weak contractiveness keeps well and fits.

Example 5.2. Let (X,d) be a metric space and $f : X \longrightarrow X$ contractive with respect to d. Consider the metric $d' := d/(1+d)$ on X. Then f is still weakly contractive with respect to d' but in general fails to stay contractive with respect to d'.

Proof. Let $0 \le K < 1$ be a Lipschitz constant for f with respect to d. Then for all $x, y \in X$

$$\begin{aligned} d'(f(x), f(y)) &= d'(x,y) - \frac{d(x,y)}{1+d(x,y)} + \frac{d(f(x),f(y))}{1+d(f(x),f(y))} \\ &\le d'(x,y) - (1-K)d(x,y). \end{aligned}$$

Since $t \mapsto t/(1+t)$ is strictly increasing $\eta(x,y) := (1-K)d(x,y)$ holds (5.3) and shows (5.4).

To complete the proof consider the function $x \mapsto x/2$, $x \in [0,\infty)$ which is contractive with respect to the metric d induced by the Euclidean norm and fails to be contractive with respect to d'. $\square$

Remark 5.2. Let (X,d) be a metric space and $A \subseteq X$ non-empty and closed. Following M. A. Krasnoselskij et. al. [KMZ+72] a function $f : A \longrightarrow X$ is said to be *generalized contractive* if there exists a function $\zeta : \mathbb{R} \times \mathbb{R} \longrightarrow [0,1)$ such that for each $0 < a \le b < \infty$

$$d(f(x), f(y)) \le \zeta(a,b) d(x,y), \quad x, y \in A, \quad a \le d(x,y) \le b.$$

It turns out that f is generalized contractive iff it is weakly contractive. Moreover weakly contractive functions have a unique fixed point. See [DG78, (3,1) Proposition] and [DG78, (1.4) Theorem], respectively, for proofs.

We will make use of the following lemma which proof can be found in [DG78, (1.3) Lemma] or [Fri96, Lemma 1.7].

Lemma 5.1. *Let (X,d) be a complete metric space, $x_0 \in X$, $r > 0$ and $f : \overline{V}_r(x_0) \longrightarrow X$ weakly contractive. For each $0 < a \le b$ let*

$$\chi(a,b) := \min\{a, \theta(a,b)\}, \tag{5.5}$$

where θ is given by (5.3), and suppose in addition that

$$d(x_0, f(x_0)) < \chi\left(\frac{r}{2}, r\right).$$

Then f has a fixed point.

Before starting the investigation whether weakly contractive functions are regular or not we derive the natural generalization of α-contractive families, see Definition 3.3. C. f. [Fri96, Definition 1.8.] for what follows.

Definition 5.2. Let (X, d) be a complete metric space and $A \subseteq X$ non-empty and closed. A homotopy $f^t : A \longrightarrow X$ is said to be a *weakly contractive family* if there exists a compactly positive function $\eta : A \times A \longrightarrow [0, \infty)$ and a continuous semi-metric d' on I such that

$$d\left(f^t(x), f^t(y)\right) \leq d(x, y) - \eta(x, y), \quad x, y \in A, \quad t \in I \tag{5.6}$$

and

$$d\left(f^t(x), f^s(x)\right) \leq d'(t, s), \quad x \in A, \quad t, s \in I. \tag{5.7}$$

Observe that every weakly contractive family f^t induces weakly contractive functions $f^{t'}$ where $t' \in I$ is fixed. Moreover we can basically follow the proof of Remark 3.1 to infer its generalization to weakly contractive families, i. e. restrictions and concatenations of weakly contractive families are weakly contractive.

Proposition 5.1. *Let $f^t : A \longrightarrow X$ be a bounded weakly contractive family. Suppose in addition $Fix((f^t)) \cap \partial A = \emptyset$. Then f^t is a regular homotopy which is A-regular on any $I' \in \mathcal{A}(I)$ where for each $t_0 \in I'$ f^{t_0} is a constant.*

Proof. First we show regularity. Observe that every weakly contractive homotopy is continuous, hence it is closed.

The proof of A-approximability follows the lines of the proof of Example 3.5.

We claim that

$$J := \left\{t \in I\,;\, x^t = f^t(x^t) \text{ for some } x^t \in \mathring{A}\right\} \tag{5.8}$$

is closed and open.

Since our definition of weak contractiveness differs from those in [DG78] we recapitulate the proof of this statement following the lines of [Fri96, Theorem 1.9].

To show that J is open let $t' \in J$ and $x' \in \mathring{A}$ such that $x' = f^{t'}(x')$. Choose $\varepsilon > 0$ such that $\overline{V}_\varepsilon(x') \subseteq \mathring{A}$ and let $\delta > 0$ be such that $d'(t', t'') < \chi(\varepsilon/2, \varepsilon)$ for $|t' - t''| < \delta$ with χ given by (5.5). Then for each such t''

$$\begin{aligned} d(x', f^{t''}(x')) &\leq d(x', f^{t'}(x')) + d(f^{t'}(x'), f^{t''}(x')) \\ &\leq 0 + d'(t', t'') \\ &\leq \chi(\varepsilon/2, \varepsilon). \end{aligned}$$

Thus, by Lemma 5.1, $f^{t''}$ has a fixed point. Hence J is open.

To show that J is closed let $x_n = f^{t_n}(x_n)$ where $t_n \in J$, $x_n \in \mathring{A}$, $n \in \mathbb{N}$ and $t_n \longrightarrow t$ for some $t \in I$. Fix $\varepsilon > 0$. Since f^t is bounded there exists $M > 0$ such that $d(x_n, x_m) \leq M$ for all $m, n \in \mathbb{N}$. Choose $n_0 \in \mathbb{N}$ such that

$$d'(t_m, t_n) < \chi(\varepsilon, M), \quad m, n \geq n_0. \tag{5.9}$$

We claim $d(x_n, x_m) < \varepsilon$ for all $m, n \geq n_0$, i. e. $(x_n)_n$ turns out to be Cauchy.

Then, since A is complete, f^t is a continuous homotopy and $Fix((f^t)) \cap \partial A = \emptyset$ by assumption, we infer $x = f^t(x)$ for some $x \in \mathring{A}$. Hence J is closed.

Assume to the contrary $d(x_m, x_n) \geq \varepsilon$ for all $m, n \geq n_0$. Then

$$\varepsilon \leq d(x_m, x_n) \leq M, \quad m, n \geq n_0 \tag{5.10}$$

and by (5.6), (5.7), and (5.9)

$$\begin{aligned} d(x_m, x_n) &= d(f^{t_m}(x_m), f^{t_n}(x_n)) \\ &\leq d(f^{t_m}(x_m), f^{t_n}(x_m)) + d(f^{t_n}(x_m), f^{t_n}(x_n)) \\ &\leq d'(t_m, t_n) + d(x_m, x_n) - \eta(x_m, x_n) \\ &< \chi(\varepsilon, M) + d(x_m, x_n) - \eta(x_m, x_n). \end{aligned}$$

Now (5.5), (5.3) and (5.10) provide the contradiction $d(x_m, x_n) < d(x_m, x_n)$.

We have shown that J of (5.8) is closed and open.

Thus $J = \emptyset$ or $J = I$ and the remaining proof of A-approximability is precisely the same as for α-contractive families since by [DG78, (1.4) Theorem] X enjoys the fixed point property for weakly contractive functions.

To complete the proof of regularity it remains to show the subnet condition.

Let $x_{\gamma'} = \big(\psi_{\gamma'}^{t_{\gamma'}} \varphi_{\gamma'}^{t_{\gamma'}}\big)(x_{\gamma'})$, $(X'_{\gamma'}, V_{\gamma'}) = \gamma' \in \Gamma'$ for some $(\psi_{\gamma'}^t, \varphi_{\gamma'}^t) \in (F^t, \gamma')$ and $\gamma' \in \Gamma' \leq \Gamma$. Switching to a subnet we can assume $(t_{\gamma'})_{\gamma'}$ is Cauchy. Moreover, since (X, d) is complete, it is sufficient to show that $(x_{\gamma'})_{\gamma'}$ is Cauchy. Assume to the contrary there exists $\varepsilon > 0$ such that for all $\gamma' \in \Gamma'$ there are $\gamma'_1, \gamma'_2 \geq \gamma'$ such that $d(x_{\gamma'_1}, x_{\gamma'_2}) \geq \varepsilon$.

Since f^t is bounded there exists, by proximity, $M > 0$ and $\gamma'_0 \in \Gamma'$ such that $d(x_{\gamma'}, x_{\gamma''}) \leq M$ for all $\gamma', \gamma'' \geq \gamma'_0$. Let

$$\delta := \min\left\{\varepsilon, \theta\left(\frac{3}{5}\varepsilon, M + \frac{2}{5}\varepsilon\right)\right\}. \tag{5.11}$$

Choose $\gamma'_1, \gamma'_2 \geq \gamma'_0$ such that

$$\varepsilon \leq d(x_{\gamma'_1}, x_{\gamma'_2}) \leq M, \quad V_{\gamma'_1} \subseteq V_{\frac{\delta}{5}},\ V_{\gamma'_2} \subseteq V_{\frac{\delta}{5}} \quad \text{and} \quad d'(t_{\gamma'_1}, t_{\gamma'_2}) < \frac{\delta}{5}. \tag{5.12}$$

Then, by proximity, there are $x'_1, x'_2 \in A$ such that

$$d(x_{\gamma'_1}, x'_1) \leq \frac{\delta}{5}, \quad d(x_{\gamma'_2}, x'_2) \leq \frac{\delta}{5} \tag{5.13}$$

and

$$\begin{aligned} d(x_{\gamma'_1}, x_{\gamma'_2}) &\leq \frac{2}{5}\delta + d\big(f^{t_{\gamma'_1}}(x'_1), f^{t_{\gamma'_2}}(x'_2)\big) \\ &\leq \frac{2}{5}\delta + d\big(f^{t_{\gamma'_1}}(x'_1), f^{t_{\gamma'_1}}(x'_2)\big) + d\big(f^{t_{\gamma'_1}}(x'_2), f^{t_{\gamma'_2}}(x'_2)\big). \end{aligned} \tag{5.14}$$

Since by (5.12), (5.13)

$$\frac{3}{5}\varepsilon \leq d(x_1', x_2') \leq M + \frac{2}{5}\varepsilon,$$

(5.6), (5.3), (5.11) provide that the second summand in (5.14) can be estimated from above by $d(x_1', x_2') - \delta$. Hence we infer from (5.14), (5.7) and (5.12)

$$\begin{aligned} d(x_{\gamma_1'}, x_{\gamma_2'}) &\leq \frac{2}{5}\delta + d(x_1', x_2') - \delta + d\big(f^{t_{\gamma_1'}}(x_2'), f^{t_{\gamma_2'}}(x_2')\big) \\ &\leq d(x_1', x_2') - \frac{3}{5}\delta + d'(t_{\gamma_1'}, t_{\gamma_2'}) \\ &< d(x_{\gamma_1'}, x_{\gamma_2'}) + \frac{2}{5}\delta - \frac{3}{5}\delta + \frac{\delta}{5} \\ &= d(x_{\gamma_1'}, x_{\gamma_2'}) \end{aligned}$$

which is a contradiction. ☐

Observe that, by Proposition 3.1, weakly contractive functions $f : A \longrightarrow X$ which have no fixed points on ∂A are regular, too.

Likewise Corollary 3.2 (b) we obtain a Nonlinear Alternative for weakly contractive families and a Sweeping theorem like Corollary 3.4 can be derived, too.

5.2 Generalizations of convexity

There is a wide variety of generalizations of convexity, see e. g. [Sol84] and [Sin97] for an overview. We pay attention to those concepts which provide a class of uniform spaces $(X, \mathcal{V}(X))$ and a class of maps $F : X \longrightarrow 2^X$ such that for each $V \in \mathcal{V}(X)$ there exists a factorization

$$\begin{tikzcd} X \arrow[rr, "VFV"] \arrow[dr, "\varphi"'] & & X \\ & P \arrow[ur, "\psi"'] & \end{tikzcd} \tag{5.15}$$

through a convex polyhedron P.

From (5.15) V-proximity immediately follows.

Recently a general discussion of such classes was given by H. Ben-El-Mechaiekh [BEM00]. In what follows we consider so called Φ-spaces which are due to C. D. Horvath [Hor91].

Let $(X, \mathcal{V}(X))$ be a uniform space and short hand for $\mathcal{A}^*(X) := \mathcal{A}(X) \setminus \{\emptyset\}$.

Definition 5.3. A map $G : \mathcal{A}^*(X) \longrightarrow 2^X$ is said to be a *c-structure* if G has contractible values and is isotonic, i. e. for each $A \in \mathcal{A}^*(X)$, $B \in \mathcal{A}^*(X)$ we have $G(A) \subseteq G(B)$ provided that $A \subseteq B$. If G is a c-structure (X, G) is said to be a *c-space.*

A c-space (X, G) is said to be a Φ-*space* if for every $V \in \mathcal{V}(X)$ there exists a map $S : X \longrightarrow 2^X$ such that

(i) $\mathcal{G}(S) \subseteq V$,

(ii) $X = \bigcup \left\{ \overset{\circ}{\widehat{S^{-1}(x)}} \,;\, x \in X \right\}$,

(iii) $G(X') \subseteq V(x)$ for all $X' \in \mathcal{A}^*(S(x))$ and all $x \in X$.

We single out one of the examples of [Hor91].

Example 5.3. Let $\alpha : I \times X \times X \longrightarrow X$ be a function such that

(a) $\alpha(0, x, y) = x$, $\alpha(1, x, y) = y$, $\quad x, y \in X$,

(b) for every $y \in X$ the function $(t, x) \mapsto \alpha(t, x, y)$ is continuous.

Then $G : \mathcal{A}^*(X) \longrightarrow 2^X$ given by

$$G(X') := \bigcap \{Y \,;\, X' \subseteq Y \subseteq X, \alpha(I \times Y \times Y) \subseteq Y\}, \quad X' \in \mathcal{A}^*(X) \tag{5.16}$$

is a c-structure and (X, G) a c-space.

Assume in addition that the uniformity is induced by a metric d and

$$d(\alpha(t, x_1, x_2), y) \leq \max\{d(x_1, y), d(x_2, y)\}, \quad x_1, x_2, y \in X, \quad t \in I. \tag{5.17}$$

Then (X, G) is a Φ-space. (Indeed, fix $V \in \mathcal{V}(X)$ and choose $\varepsilon > 0$ such that $V_\varepsilon \subseteq V$. Then $S := V_\varepsilon$ holds (i), (ii) and (iii) follows from (5.17).)

Example 5.4. Let $(X, +, d)$ be a metric topological group. Assume there exists a continuous homotopy $h^t : X \longrightarrow X$ such that $h^0 = 0$ and $h^1 = id$. Then $\alpha : I \times X \times X \longrightarrow X$ given by $\alpha(t, x, y) := h^{1-t}(x) + h^t(y)$, $x, y \in X$, $t \in I$ fulfills (a) and (b) of Example 5.3. and (X, G) is a c-space where G is defined as in (5.16).

Suppose in addition

$$d\left(h^{1-t}(x_1) + h^t(x_2), y\right) \leq \max\{d(x_1, y), d(x_2, y)\}, \quad x_1, x_2, y \in X, \quad t \in I. \tag{5.18}$$

Then (X, G) is a Φ-space.

E. g. normed vector spaces X are Φ-spaces: define $h^t(x) := tx$, $x \in X$, $t \in I$ and it turns out that G is the *convex hull operator*, i. e. $G(Y) = conv(Y)$, $Y \subseteq X$, which is apparently a c-structure.

The crucial quality of the class of Φ-spaces turns out to be that for the class of compact functions there exist factorizations like (5.15) where P is a simplex. Due to our need of preciseness we present a statement which is slightly different from [Hor91]. We give its proof for the sake of completeness.

Lemma 5.2. *Let (X, G) be a Φ-space such that $x \in G(\{x\})$ for every $x \in X$. Let $Y \subseteq X$ be non-empty and compact. Then for every $V \in \mathcal{V}(X)$ and every $Y' \in \mathcal{A}(Y)$ there exists a selection of $V|_Y$ which factorizes continuously through a simplex*

$$\begin{tikzcd} Y \arrow[rr, "V|_Y"] \arrow[dr, "\eta"'] & & X \\ & \Delta_n \arrow[ur, "\zeta"'] & \end{tikzcd} \tag{5.19}$$

such that $(\zeta\eta)(y') = y'$ for each $y' \in Y'$.

Proof. Choose $V' \in \mathcal{V}(X)$ such that $V'V' \subseteq V$. Since X is a Φ-space there exists a map $S : X \longrightarrow 2^X$ which holds (i) to (iii) of Definition 5.3 with respect to V' instead of V of Definition 5.3. Since Y is compact we infer from (ii) the existence of $x_1, \ldots, x_m \in X$ such that, using the notation

$$X_i := \overset{\circ}{\widehat{S^{-1}(x_i)}}, \quad i = 1, \ldots, m, \tag{5.20}$$

$\{X_1, \ldots, X_m\}$ covers Y. Let $\{O_1, \ldots, O_n\}$ be a point-star-refinement of $\{Y \cap X_1, \ldots, Y \cap X_m\}$ such that each $y' \in Y'$ belongs to one and only one O_i.

It is sufficient to show the existence of a continuous selection $g : Y \longrightarrow X$ of the map $(V'V')|_Y$ which is the identity on Y' and factorizes through a simplex like

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "\eta"'] & & X \\ & \Delta_n \arrow[ur, "\zeta"'] & \end{tikzcd} \tag{5.21}$$

Identify $\{O_1, \ldots, O_n\}$ with the vertices of the simplex Δ_n. We construct $\zeta : \Delta_n \longrightarrow X$ inductively by defining $\zeta_k = \zeta|_{\Delta_n^k}$ on each k-skeleton Δ_n^k of Δ_n.

To define ζ_0 on Δ_n^0 choose $y_i \in O_i$ for each vertex O_i such that $y_i = y'$ if $y' \in O_i$. This is possible since each O_i contains at most one y'. Observe that ζ_0 is continuous and holds

$$\zeta_0(\sigma) = y_i \in G(\{y_i\}) = G(\{y_i\,;\, O_i \in \sigma\}), \quad \sigma \in \Delta_n^0 \tag{5.22}$$

since $x \in G(\{x\})$ for any $x \in X$.

Let $k \in \{1, \ldots, n\}$ and assume $\zeta_{k-1} : \Delta_n^{k-1} \longrightarrow X$ holds

$$\zeta_{k-1}(\sigma) \subseteq G(\{y_i\,;\, O_i \in \sigma\}), \quad \sigma \in \Delta_n^{k-1}. \tag{5.23}$$

To define ζ_k on Δ_n^k fix $\sigma \in \Delta_n^k \setminus \Delta_n^{k-1}$. Since G is a c-structure by (5.23)

$$\begin{aligned} \zeta_{k-1}(\partial\sigma) &= \bigcup \left\{ G\left(\{y_i\,;\, O_i \in \tilde{\sigma}\}\right) \,;\, \tilde{\sigma} \in \partial\sigma \right\} \\ &\subseteq G\left(\{y_i\,;\, O_i \in \tilde{\sigma}, \tilde{\sigma} \in \partial\sigma\}\right) \end{aligned}$$

and the latter set is a contractible space. By Lemma 1.3 we can extend $\zeta_{k-1}|_{\partial\sigma}$ continuously over σ to $\zeta_k|_\sigma : \sigma \longrightarrow G(\{y_i\,;\, O_i \in \sigma\})$.

Proceeding in the same manner for all simplices $\sigma \in \Delta_n^k \setminus \Delta_n^{k-1}$ we obtain a continuous extension ζ_k of ζ_{k-1} over the k-skeleton Δ_n^k. Moreover, by (5.23) and our construction,

$$\zeta_k(\sigma) \subseteq G\left(\{y_i\,;\, O_i \in \sigma\}\right), \quad \sigma \in \Delta_n^k. \tag{5.24}$$

Let $\mathcal{S}$ be the nerve of the covering $\{O_1, \dots, O_n\}$, $b : Y \longrightarrow \mathcal{S}$ a barycentric function (with respect to $\{O_1, \dots, O_n\}$) and $\iota : \mathcal{S} \hookrightarrow \Delta_n$ the embedding.

Define $\zeta := \zeta_n$, $\eta := \iota b$ and $g := \zeta\eta$.

Since b is a barycentric function with respect to $\{O_1, \dots, O_n\}$, by (5.24) and isotonicity of G, for any $x \in Y$

$$\begin{aligned} g(x) &= (\zeta_n b)(x) \\ &= \bigcup \{\zeta_n(\sigma)\,;\, \sigma \in \mathcal{S}, O_i \in \sigma, x \in O_i\} \\ &= G(\{y_i\,;\, x \in O_i\}). \end{aligned} \tag{5.25}$$

Since $\{O_1, \dots, O_n\}$ is a point-star-refinement of $\{X_1, \dots, X_m\}$ there exists by (5.20) and index $i_0 \in \{1, \dots, m\}$ such that $O_i \in S^{-1}(x_{i_0})$ for all O_i of (5.25). Hence $g(x)$ belongs to $G(\{y_i\,;\, O_i \subseteq S^{-1}(x_{i_0}), x \in S^{-1}(x_{i_0})\})$. Since the y_i are elements of the O_i we conclude by (iii) and (i) of Definition 5.3 $g(x) \in V'(x_{i_0})$ and $(x, x_{i_0}) \in V'$ which implies $g(x) \in V(x)$.

Moreover if $x = y'$ for some $y' \in Y'$ then y' belongs to one and only one of the O_i. Hence for each $y' \in Y'$ there exists O_{i_0} such that $g(y') = (\zeta_n b)(y') = \zeta_0(\{O_{i_0}\}) = y'$ by (5.22). I. e. g is the identity on Y'. $\square$

Proposition 5.2. *Let (X, G) be a Φ-space such that $x \in G(\{x\})$ for each $x \in X$. Let $A \subseteq X$ be non-empty and closed and Let $f^t : A \longrightarrow X$ a compact continuous homotopy. Then f^t is A-regular on any $I' \in \mathcal{A}(I)$ where for each $t' \in I'$ $f^{t'}$ is uniformly continuous.*

Proof. Since f^t is a compact homotopy it is sufficient to show approximability of f^t. Let $(X', V) = \gamma \in \Gamma(X)$. Let $Y := \overline{\tilde{f}(A)}$, where $\tilde{f}$ is associated to the homotopy f^t. Then Y is compact. We claim the upper triangle of

$$\begin{tikzcd} A \arrow[rr, "f^t"] \arrow[dr, "\eta f^t"'] \arrow[dd, "f^t"'] & \arrow[d, phantom, "\gamma"] & X \\ & \Delta_n \arrow[ur, "id\,\zeta"'] \arrow[dr, "\zeta"] & \\ Y \arrow[ur, "\eta"] \arrow[rr, "V|_Y"] & & X \arrow[uu, "id"'] \end{tikzcd} \tag{5.26}$$

where the lower triangle comes from Lemma 5.2 applied to V and $Y' = X' \cap Y$.

In fact, Δ_n is a convex polyhedron and $(id\,\zeta)$, (ηf^t) are continuous homotopies. Fix $t \in I$. From $X' \cap Fix(f^t) \subseteq Y'$ we infer for any $x' \in X' \cap Fix(f^t)$

$((id\,\zeta)(\eta f^t))(x') = (\zeta\eta)(x') = x'$ since $(\zeta\eta)$ is the identity on Y'. Since t is arbitrary this is preciseness on X'.

The factorizations (5.26) provide $\mathcal{G}((id\,\zeta)(\eta f^t)) \subseteq \mathcal{G}(Vf^t) \subseteq \mathcal{G}(VF^tV_A)$. which is A-approximability of f^t.

Apparently A-regularity of f^t holds on any $I' \in \mathcal{A}(I)$ where for each $t' \in I'$ $f^{t'}$ is uniformly continuous. □

We infer from Proposition 3.1 that under the hypothesis of Proposition 5.2 any compact continuous function $f : A \longrightarrow X$ turns out to be A-regular.

From Corollary 3.2 (a) and Proposition 5.2 we obtain a Nonlinear Alternative as follows.

Theorem 5.1. *Let (X, G) be a contractible and locally contractible metrizable Φ-space such that $x \in G(\{x\})$ for each $x \in X$. Let $A \subseteq X$ be closed. Let $f^t : A \longrightarrow X$ be a compact continuous homotopy such that $f^1|_{\partial A} = x_0$ for some $x_0 \in \mathring{A}$. Then at least one of the following statements holds:*

(i) f^0 has a fixed point.

(ii) There exists $0 < t < 1$ such that f^t has a fixed point on ∂A.

Again we emphasize that Theorem 5.1 is well-known even for general Φ-spaces since by [Hor91, Theorem 4] every Φ-space is a fixed point space for the class of compact continuous functions. Hence Theorem 5.1 follows by means of classical Topological Transversality, see [Gra01].

Finally Example 5.4 and Theorem 3.4 provide the following Sweeping theorem. Observe that by (5.18) $G(\{x\}) = \{x\}$ for all $x \in X$ where G is the c-structure of Example 5.4. Recall now our discussion of compact homotopies following the proof of Theorem 3.4 to obatin

Theorem 5.2. *Let $(X, +, d)$ be a complete metric topological group which is locally contractible. Assume there exists a continuous homotopy $h^t : X \longrightarrow X$ such that $h^0 = 0$, $h^1 = id$ and (5.18) hold. Let $f^t : X \longrightarrow X$ be a compact continuous homotopy.*

Then $\{a, b\}$ is swept by $(id - f^t)(A)$ provided that a and b belong to the same component of $X \setminus (id - f^1)(A)$ and to different components of $X \setminus A$.

(i) to (iii) of Definition 5.3 can be understand as a substitute for local convexity. Fréchet spaces generalizes now as follows. C. f. [Hor91] and [Hor93] where the following definitions are given in full generality.

Definition 5.4. Let (X, G) be a c-space. A subset $Y \subseteq X$ is said to be a *G-set* if $G(\mathcal{A}^*(Y)) \subseteq Y$.

(X, G) is said to be an *l.c.-space* if there exists a base $\mathcal{V}(X)$ for the uniformity of X such that $V(Y)$ is a G-set provided that Y is a G-set and $V \in \mathcal{V}(X)$.

Suppose X is in addition metrizable. Then (X, G) is said to be an *m.c.-space* provided that every point in X has a neighborhood base consisting of G-sets.

Let (X, G, d) be a metric space. (X, G, d) is said to be a *metric l.c.-space* provided that open balls $V_r(x)$ are G-sets and $V_r(Y)$ is a G-set whenever Y is a G-set.

Observe that in m.c.-spaces singletons are G-sets, i. e.

$$G(\{x\}) = \{x\}, \quad x \in X. \tag{5.27}$$

Indeed, let (X, G) be an m.c.-space and d a metric which induces the topology for X. Let $x \in X$. Then, since G is isotonic, $G(\{x\}) \subseteq \cap_{r>0} G\big(\mathcal{A}^*(V_r(x))\big)$. Since x has a neighborhood base of G-sets the latter set is contained in $\cap_{r>0} V_r(x)$ which is the singleton $\{x\}$. Hence, since $G(\{x\})$ is non-empty, $G(\{x\}) = \{x\}$.

Apparently every metric l.c.-space is an m.c.-space. Moreover every m.c.-space is a Φ-space. Indeed, if (X, G) is an m.c.-space and d a metric which induces the topology of X then for any $x \in X$ and any $\varepsilon > 0$ there exists $\delta = \delta(x, \varepsilon) > 0$ and a G-set $B = B(x, \varepsilon)$ such that $V_\delta \subseteq B \subseteq V_\varepsilon$. Fix $V \in \mathcal{V}(X)$ as in Definition 5.3 and choose $\varepsilon > 0$ such that $V_\varepsilon \subseteq V$. Then $S(x) := V_{\delta(x,\varepsilon)}$, $x \in X$ fulfills the hypothesis of Definition 5.3.

The following example of a metric l.c.-space is from [Hor91].

Example 5.5. Let (X, d) be a complete metric space and α a function as in Example 5.3. Assume in addition that

$$d(\alpha(t, x_1, x_2), \alpha(t, y_1, y_2)) \leq \max\{d(x_1, y_1), d(x_2, y_2)\}, \quad x_1, x_2, y_1, y_2 \in X, \quad t \in I. \tag{5.28}$$

Then (X, G, d) is a complete metric l.c.-space where G is given by (5.16).

Further examples of metric l.c.-spaces come from the classes of metrizable *normally supercompact spaces* and *hyperconvex spaces*, respectively. We refer to [Hor93] for the definitions of these classes and for the investigation regarding metric l.c.-spaces.

Remark 5.3. We already noticed that Examples 5.3 and 5.4 can be motivated by normed vector spaces. Since moreover (5.28) holds if the metric d is induced by a norm the question arises how far we are away from normed linear spaces.

Let (X, d) be a metric space and consider for fixed $x_1, x_2 \in X$ solutions x of the variational inequalities

$$d(x, y) \leq (1-t)d(x_1, y) + td(x_2, y), \quad y \in X \quad \text{if} \quad t \in I, \tag{5.29}$$

$$d(x, y) \geq (1-t)d(x_1, y) + td(x_2, y), \quad y \in X \quad \text{if} \quad t \in \mathbb{R} \setminus I. \tag{5.30}$$

S. Gähler and G. Murphy [GM81] showed that in normed linear spaces $x = (1-t)x_1 + tx_2$ is the unique solution of (5.29) and (5.30), respectively, and they investigated necessary and sufficient conditions of the solution sets of (5.29) and (5.30) such that there exists the algebraic structure of a linear space on X and d turns out to be induced by a norm.

W. Takahashi [Tak70] considered metric spaces (X, d) where the solution sets of (5.29) are non-empty. He interpreted a fixed selection $\alpha(t, x_1, x_2) = x$ as abstract convex combination of x_1 and x_2 and generalized the fixed point theorem of W. A. Kirk [Kir65] for non-expansive functions. This development was continued by several authors, see e. g. [GSW82], [Tal77] and [BA96].

Based on this notion of abstract convexity convex- and affine functionals can be defined and it turns out that if the affine functionals separate the points of X we can embed X affinely into a locally convex space, see e. g. [Oko00b, Theorem 2].

Moreover a notion of extreme point can be defined, too, and, provided (X, d) is of negative curvature, i. e. the chosen solutions $\alpha(t, x_1, x_2)$ of (5.29) hold the following, stronger version of (5.28)

$$d(\alpha(t, x_1, x_2), \alpha(t, y_1, y_2)) \leq (1 - t)d(x_1, y_1) + td(x_2, y_2), \quad x_1, x_2, y_1, y_2 \in X, \quad t \in I, \tag{5.31}$$

a Krein-Milman theorem can be derived, see [Oko00a, Theorem 7].

A metric space (X, d) is said to be *convex in the metric sense* provided for any two different points a and b in X there exists a point c in X, different from a and b, which is between a and b, i. e. $d(a, b) = d(a, c) + d(b, c)$.

It turns out that solutions x of (5.29) are between x_1 and x_2 provided $0 < t < 1$. See [GM81] or [Oko00a] for a proof. Hence X is convex in the metric sense and, by a result of K. Menger, see [Rin61], we are concerned with spaces with inner metric whenever (X, d) is complete.

Examples of spaces of negative curvature are complete simple connected Riemannian manifolds with geodesic metric which are of non-positive sectional curvature, consider e.g. the hyperbolic plane, see [Bus48], [GM81] and [Rin61].

[Hor91, Theorem 3] provides an analogue of Michael's selection theorem for complete metric l.c.-spaces and lower semicontinuous maps with non-empty closed G-sets as values. Recall that, due to the need of preciseness, we gave a generalized version of the classical Michael selection theorem, see Theorem 1.1. Since this generalization only makes additional use of the lower semicontinuity we are in position to modify [Hor91, Theorem 3.] in the same manner whenever (5.27) holds, i. e. singletons are G-sets. The proof of the following proposition follows therefore straightforwardly from Proposition 5.2. C. f. with [Hor93, Corollary], too.

Proposition 5.3. *Let (X, G) be a complete metric l.c.-space. Let $A \subseteq X$ be non-empty and closed and $F^t : A \longrightarrow 2^X$ a compact lower semicontinuous closed homotopy with non-empty G-sets as values. Then F^t is A-regular in all $I' \in \mathcal{A}(I)$ where for each $t' \in I'$ $F^{t'}$ is uniformly upper semicontinuous.*

The formulation of the corresponding Nonlinear Alternative is obvious and well known. Moreover a Sweeping theorem can be derived since in view of (5.27) any arc can be considered as a closed lower semicontinuous homotopy with G-sets as values.

From [Hor93, Theorem 6.] we obtain an uniform approximation result for compact metric l.c.-spaces and closed homotopies which have G-sets as values. Since in metric l.c.-spaces singletons are G-sets a slight modification of the proof of [Hor93, Theorem 6.] allows us to show preciseness for this approximations. We omit the proof of

Proposition 5.4. *Let (X, G) be a compact metric l.c.-space. Let $A \subseteq X$ be non-empty and closed and $F^t : A \longrightarrow 2^X$ a closed homotopy with non-empty G-sets as values. Then F^t is A-regular.*

5.3 Some interconnections

We illustrated regularity by quite a lot of examples so far and the question arises how much redundance there is. Spaces with simplicial approximation property are our major example where - up to the best knowledge of the author - classical approximation techniques fail and our concept of approximability comes into operation.

Primary we are forced to put the question whether the Roberts spaces are AR(compact metric). As convex subsets of HTVS Roberts spaces are contractible (and locally contractible) and in view of Theorem 1.7 the question reduces to ANR(compact metric).

Nguyen To Nhu et. al. [Nhu97], [NSA97a], [NSA97b] investigated the AR-property of Roberts spaces and it turns out, see [NSA97a, Theorem 2.], that on conditions regarding cardinality and linear independence of the involved needle-sets Roberts spaces are AR(compact metric). However, for Robert's original example [Rob77a] and the spaces K_R, constructed in Proposition 4.3, it seems to be still an open problem whether these spaces enjoy AR(compact metric) or not. Moreover it is unknown whether closed convex subsets of the Roberts spaces are AR(compact metric).

Up to the best knowledge of the author it is moreover an open problem whether there exists a compact convex subset of a metrizable HTVS which in not AR(compact metric). In absense of compactness the AR-problem was solved by a celebrated result of R. Cauty [Cau94] who shows the existence of a (non compact) metrizable linear σ-compact space which is not AR(metric) (see Definition 5.5 for the definition of σ-compactness). Cauty's result was moreover the first example of a HTVS which is not admissible. C. f. Theorem 5.3 for this interconnection.

In what follows we state some relations between absolute retract/extension-properties, admissibility, c-structures and regularity.

Consider the class of all non-empty convex subsets K of HTVS and their subclasses consisting of

(a) admissible spaces in the sense of Definition 4.3,

(α) complete metrizable admissible spaces,

(b) complete metric l.c.-spaces (K, G, d) in the sense of Definition 5.4,

(b') m.c.-spaces (K, G) in the sense of Definition 5.4,

(β) Φ-spaces (K, G) in the sense of Definition 5.3 where $x \in G(\{x\})$,

(γ) AR(metric),

(γ') metrizable unions $K = \cup_{i=1}^{\infty} K_i$ of closed AE(compact metric) K_i such that $K_i \subseteq \mathring{K}_{i+1}$, $i \in \mathbb{N}$,

(δ) spaces K such that $f : A \longrightarrow K$ is A-regular whenever $A \subseteq K$ is non-empty, compact and f continuous,

respectively. So far we know the inclusions

$$\begin{tikzcd}
 & (\alpha) \arrow[r] \arrow[dl] \arrow[d] \arrow[dr] & (a) \arrow[d] \\
(\gamma) \arrow[r] & (\gamma') \arrow[r] & (\delta) \\
(b) \arrow[r] \arrow[u] & (b') \arrow[r] \arrow[ul] \arrow[u] \arrow[ur] & (\beta) \arrow[u]
\end{tikzcd} \tag{5.32}$$

In fact, we already know the horizontal arrows: $(\alpha) \to (a)$ is obvious, $(b) \to (b') \to (\beta)$ was discussed following Definition 5.4, $(\gamma) \to (\gamma')$ follows from Proposition 1.1 (iii) and $(\gamma') \to (\delta)$ from Remark 2.2.

Furthermore $(\alpha) \to (\gamma)$ is a result of V. Klee [Kle60b, Theorem. 10] and $(b') \to (\gamma)$ is a result of C. D. Horvath [Hor93].

$(\beta) \to (\delta)$ follows from Proposition 5.2.

$(a) \to (\delta)$ was not shown explicitly so far. Fix a space $K \in (a)$ and consider the proof of Theorem 1.4. It carries through for convex subsets of non-locally convex spaces, too, and we obtain a suitable analogue of our customized form of Schauder's projection. Thus $K \in (\delta)$, since regularity of K is equivalent to approximability which itself follows now likewise Example 2.3.

An elaborated discussion of the interconnections of the classes $(\alpha), (\gamma)$ and (γ') can be found in [vdBvdM88] and [vdBDHvdM92].

For the following definition see [Dug73, Chap. KI, 7.].

Definition 5.5. A locally compact space K is said to be *σ-compact* if it fulfills the following equivalent hypothesis:

(i) K is a Lindelöf space,

(ii) K is the union of at most countable many compact spaces,

(iii) $K = \cup_{i=1}^{\infty} K_i$ with compact K_i such that $K_i \subseteq \mathring{K}_{i+1}$, $i \in \mathbb{N}$.

Observe that every σ-compact space is a complete Tychonoff space.

For σ-compact spaces the greek part of diagram (5.32) collapses.

Theorem 5.3. *For the class of metrizable σ-compact convex subsets of HTVS the subclasses (α), (β), (γ), (γ') and (δ) coincide.*

Proof. By [Dob85, Corollary 1] the classes (α), (γ) and (γ') coincide and by (5.32) it is sufficient to show $(\delta) \to (\alpha) \to (\beta)$.

To see $(\delta) \to (\alpha)$ fix K' and $V \in \mathcal{V}(K)$ as in Definition 4.3. Choose $V' \in \mathcal{V}(K)$ such that $V'V' \subseteq V$. Since $\iota : K' \hookrightarrow K$ is assumed to be K'-regular there exists a selection

$$\begin{tikzcd} K' \arrow[rr, "V'|_{K'}"] \arrow[dr, "\varphi"'] & & K \\ & P \arrow[ur, "\psi"'] & \end{tikzcd}$$

through a polyhedron P. Since P is not necessarily contained in K we take a second triangle from Lemma 4.1 to infer

$$\begin{tikzcd} K' \arrow[rr, "V'V'|_{K'}"] \arrow[dr, "\varphi"'] & & K & \\ & P \arrow[ur, "V'\psi"] \arrow[rr, "\psi'"] & & P' \arrow[ul, hook', "\iota"'] \end{tikzcd}$$

Then $I' := \psi'\varphi$ fulfills the requirements of Definition 4.3.

To see $(\alpha) \to (\beta)$ let $K = \cup_{i=1}^{\infty} K_i$ be a covering of K as in (iii) of Definition 5.5. Fix an open $V \in \mathcal{V}(K)$ and choose for each K_i an approximation $I_i' : K_i \longrightarrow K$ as in Definition 4.3. Since $(I_i', id)(K_i) \subseteq V$, V is open and all I_i' map to finite dimensional linear subspaces E_i of the underlying HTVS there exist Euclidean balls $V_\varepsilon^i(0)$ in E_i such that

$$K \cap (V_\varepsilon^i I_i')(x) \in V(x), \quad x \in K_i \tag{5.33}$$

for all i. For any $x \in K$ let $i(x) := \min\{i \in \mathbb{N};\, x \in \mathring{K}_i\}$. Define $S : K \longrightarrow 2^K$ by

$$S(x) := K \cap \left(V_\varepsilon^{i(x)} I_{i(x)}'\right)(x), \quad x \in K \tag{5.34}$$

and let $G : K \longrightarrow K$ be the convex hull operator.

We claim (i) to (iii) of Definition 5.3. In fact, $\mathcal{G}(S) \subseteq V$ follows directly from (5.34) and (5.33). (iii) follows since S is convex-valued, and G is the convex hull operator. (ii) holds since $x \mapsto i(x)$ is locally constant and therefore each $x \in K$ has a neighborhood K_x such that $S|_{K_x}$ has an open graph in $K_x \times E_i$. $\square$

We point out that the above classes do not coincide with the class of all metrizable σ-compact convex subsets of HTVS since there exists a σ-compact space which is not AR(metric). See R. Cauty [Cau94].

6 Concluding remarks and perspectives

We list some open questions which have not been under consideration so far.

Acyclicity. We have not considered maps where fixed point results base on the Vietoris-Begle mapping theorem, see [Gór76, (4.2)]. E. g. Theorem 1.14, the Eilenberg Montgomery fixed point theorem [EM46], is of that type. [Gór76] gives a survey of this direction. There exist approximation/selection results for these maps which are equivalent to the proximity part of our A-approximability. See e. g. [GGK91], [Kry94], [Bad96] and the references therein. Hence we have to put the question what about preciseness of this approximations.

Simple arcs. Proposition 2.5 and Corollary 2.4 prompt the question whether or not it is possible to generalize their statement to homotopies $F^t : A \longrightarrow 2^X$. The definition of triangles claims by (ii^t) and (iii^t) a 'parametrized control' of $Fix((F^t))$. Hence we are forced to consider continuous deformations of simple arcs in X and the question arises what kind of regular homotopies, in the sense of Definition 3.2, can be characterized by those deformations.

This problem is closely related to the question whether convex polyhedra are the right candidates for the lower tips of our triangles.

Continua. Our results regarding Topological Transversality take place in metrizable contractible and locally contractible spaces X. From an additional supposed compactness of X the subnet condition becomes superfluous. Then X is a compact, connected and locally connected space and, by the Hahn-Mazurkiewicz theorem, see e. g. [Sag94, (6.8) Theorem], a continuous image of the unit interval I. Hence fixed point theorems for continua come to mind and results regarding Topological Transversality again should be closely related to Proposition 2.5. See moreover [Hag91] and [Hag98] for fixed point results regarding functions that are deformations, i. e. functions which are homotopic to the identity.

Odd functions. The formulation of our Sweeping Theorem needs group-structure and the question arises what about essentiality of odd functions.

Recall that a subset A of a group $(G, +)$ is said to be *symmetric* provided $x \in A$ iff $-x \in A$. A function $f : A \longrightarrow G$ is said to be *odd* (or *antipodal preserving*) provided $f(-x) = -f(x)$, $x \in A$.

Consider the cubes I^n and their boundaries S^{n-1} in $\mathbb{R}^n$. By Theorem 1.9 any continuous odd function $f : S^n \longrightarrow \mathbb{R}^n$ is essential (in the sense of [DG82, (4.3) Definition]) with respect to I^n, i. e. every continuous extension $\hat{f} : I^n \longrightarrow \mathbb{R}^n$ of f has a fixed point.

The proof that constant functions are essential in our sense, Theorem 3.1, relies basically on our 'cone-technique' in the proof of Theorem 2.5 which makes fundamental use of Lemma 1.2 and Lemma 1.3. Both Lemmata are local in nature and, at a first glance, it seems impossible to use a corresponding technique appropriate to the global quality of preserving antipodal points.

Measures of non-compactness. Corollary 3.2 classically applies to Fréchet spaces X, subsets $A = \overline{U}$ where U is open, bounded and contains the origin 0, and homotopies $F^t = (1-t)F$, $t \in I$ where $F : \overline{U} \longrightarrow 2^X$ is closed, compact and convex-valued or $F = f$ is single-valued and contractive, respectively.

It is natural to ask whether this classical Nonlinear Alternative holds for convex combinations $\lambda F + (1-\lambda) f$ with compact F, contractive f and $\lambda \in I$. The answer is affirmative and leads to measures of non-compactness and condensing homotopies.

We consider the Hausdorff measure of non-compactness and follow [Sad72], [Jer82].

Let X be a Fréchet space. For any subset Y of X

$$\chi(Y) := \inf\{\rho > 0\,;\, Y \text{ admits a finite } \rho\text{-mesh}\}$$

is called the *Hausdorff measure of non-compactness* of the set Y. χ defines an isotonic functional from the power set of X to $[0, \infty]$ which vanishes iff Y is precompact and holds $\chi(\overline{conv\, Y}) = \chi(Y) = \chi(Y \cup \{y\})$ for any $y \in X$.

Let $A \subseteq X$ be non-empty and closed, $F^t : A \longrightarrow 2^X$ a homotopy and $k \in [0, \infty)$. F^t is said to be *k-condensing* provided the associated map $\tilde{F}$ holds

$$\chi(\tilde{F}(I \times Y)) \leq k\chi(Y), \quad Y \subseteq X.$$

Suppose now $F^t : A \longrightarrow 2^X$ is a closed convex-valued upper semicontinuous (metrically) bounded homotopy which is k-condensing with respect to $0 \leq k < 1$. Then the descending sequence

$$\begin{aligned} X_0 &:= X, \\ X_n &:= \overline{conv}(\tilde{F}(I \times (A \cap X_{n-1}))), \quad n \in \mathbb{N} \end{aligned} \tag{6.1}$$

stabilizes in a non-empty compact convex subset X_∞ of X which is called the *limit range* of F^t. X_∞ is $AR(metric)$ by the Dugundji theorem, $F^t(A \cap X_\infty) \subseteq X_\infty$ and $Fix((F^t)) \subseteq X_\infty$.

Fixed point theorems and a Nonlinear Alternative are well-known for k-condensing homotopies, see e. g. [Kay74] and [Jer82, Chapter 3].

We sketch now that it is also possible to apply Theorem 3.3 to infer a Nonlinear Alternative.

Consider a homotopy $F^t : A \longrightarrow X$ like above such that $F^1|_{\partial A} = x_0$ for some $x_0 \in \mathring{A}$. The topological hypothesis of Theorem 3.3 are apparently fulfilled.

To show approximability fix $\gamma \in \Gamma(X)$. Adjoin in (6.1) the point x_0 to $\tilde{F}(I \times (A \cap X_{n-1}))$. Since χ is invariant with respect to building the convex hull the new X_∞ keeps being compact and contains x_0. Choose an appropriate Schauder projection s which is precise on $X' \cup \{x_0\}$ and apply Lemma 1.1 to $\tilde{F}|_{I\times(A\cap X_\infty)}$ to obtain a continuous homotopy $f^t_\infty : A \cap X_\infty \longrightarrow X_\infty$ and a triangle

$$\begin{tikzcd}
A \cap X_\infty \arrow[rr, "F^t"] \arrow[d, "f^t_\infty"'] \arrow[dr, "sf^t_\infty"] & \arrow[d, phantom, "\gamma"] & X_\infty \\
X_\infty \arrow[r, "s"] & P \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$

such that, in addition, $f^1_\infty|_{\partial A\cap X_\infty} = x_0$. By the Dugundji theorem (sf^t_∞) can be extended to a continuous homotopy $f^t : A \longrightarrow P$ subject to $f^1|_{\partial A} = x_0$. We obtain

$$\begin{tikzcd}
A \cap X \arrow[rr, "F^t"] \arrow[dr, "f^t"] & \arrow[d, phantom, "\gamma"] & X \\
X_\infty \arrow[u, hook] \arrow[r, "sf^t_\infty"] & P \arrow[ur, hook, "\iota"'] \arrow[r, hook, "\iota"] & X_\infty \arrow[u, hook]
\end{tikzcd}$$

To see the subnet condition let $x_{\gamma'} = \left(\psi^{t_{\gamma'}}_{\gamma'} \varphi^{t_{\gamma'}}_{\gamma'}\right)(x_{\gamma'})$ where $(\psi^t_{\gamma'}, \varphi^t_{\gamma'}) \in (F^t, \gamma')$ and $\gamma' \in \Gamma'$ for some $\Gamma' \leq \Gamma$. Since f^t is k-condensing and $0 \leq k < 1$ we infer, similar as in the proof of the subnet condition for α-contractive homotopies, see Example 3.5, for every $\gamma' \in \Gamma'$ and every $\varepsilon > 0$ a $\gamma'' \geq \gamma'$ and a finite number of ε-balls that cover $\{x_{\gamma'}\,;\, \gamma' \geq \gamma''\}$. Since at least one of these balls must contain a cofinal subnet of $(x_{\gamma'})_{\gamma'\geq\gamma''}$ we obtain a Cauchy subnet $(x_{\gamma''})_{\gamma''}$ of $(x_{\gamma'})_{\gamma'}$ which converges by means of completeness of X.

Hence F^t is regular.

Since we can consider F^0 as a k-bounded (constant) homotopy F^0 is regular, too. ∂A-regularity of F^t in $t = 1$ follows from $f^t|_{\partial A} = x_0$ and since $F^1 = x_0$ is uniformly upper semicontinuous as a constant function.

Observe finally that can we get rid of the V in Theorem 3.3 by means of a similar argumentation as for the proof of the subnet condition.

So far there is nothing new. Observe now that we neither need X_∞ *is* AR(*metric*) nor we need $F^t(A\cap X_\infty) \subseteq X_\infty$ or upper semicontinuity of $\tilde{F}$ for the proof of regularity of $F^t : A \longrightarrow 2^X$. In fact, it is sufficient to suppose that there exists a closed subset Y of X such that

(i) $Fix((F^t)) \subseteq Y$,

(ii) Y is AE(finite dimensional metric),

(iii) $F^t|_{A\cap Y} : A \cap Y \longrightarrow 2^X$ is regular.

From this approximability can be derived. Moreover the subnet condition relies only on the metric qualities of the Hausdorff measure of non-compactness χ.

However, at first sight there is no natural condition on that (∂A)-regularity of $F^t|_{A\cap Y}$ implies those of F^t.

Spaces with simplicial approximation property. By Theorem 5.3 any metrizable compact convex subset K of a HTVS is admissible iff every continuous $f : A \longrightarrow K$ is A-regular. By Proposition 4.1 every continuous function $f : A \longrightarrow K$ is $\emptyset$-regular provided K is a space with simplicial approximation property. Hence the question arises whether it is possible to characterize the simplicial approximation property in terms of B-regularity for B belonging to some subclass of the class of closed subsets of K.

Moreover, up to the best knowledge of the author, it is an open problem whether compact convex subsets of R. Cauty's [Cau94] example of a non-admissible space are spaces with simplicial approximation property or not.

The F-space sampler [KPR84, Chapter 9,4.] states more open questions with regard to spaces with simplicial approximation property.

The index. Consider a regular map $F : A \longrightarrow 2^X$. Approximability and the subnet condition basically put us in a position to separate the demands for the existence of approximative fixed points $x_\gamma = (\psi_\gamma\varphi_\gamma)(x_\gamma)$, $\gamma \in \Gamma$ and for the existence of at least one cluster point of $(x_\gamma)_\gamma$ which belongs to $Fix(F)$. Up to the best knowledge of the author this idea is due to F. E. Browder and W. V. Petryshyn [BP68] which defined a *topological degree* for *A-proper* functions. See moreover [Pet93] and the references therein. Motivated by this the author [Oko95] considered a topological degree for *approximation-compact* functions in admissible (in the sense of Definition 4.3) HTVS. See moreover [Kry94, Chapter III-V] for a general concept based on *filtrations*.

All these concept have in common that it is possible to define a topological index/degree by means of the cluster points of the indexes/degrees of the approximating functions $f_\lambda : X_\lambda \longrightarrow X_\lambda$, $\lambda \in \Lambda$ where Λ is a directed set. The index/degree of $f_\lambda : X_\lambda \longrightarrow X_\lambda$ is well-defined and independent of the concrete candidate f_λ for the approximation on X_λ provided $\lambda \geq \lambda_0$ for some suitable chosen $\lambda_0 \in \Lambda$. More precisely there always exists a separation property which allows to connect these candidates with a homotopy $f^t_\lambda : X_\lambda \longrightarrow X_\lambda$ on which the index/degree is constant.

Proposition 2.6 provides in (2.10) such a separation property. In fact, consider a open subset U of a Tychonoff space X and a regular map $F : \overline{U} \longrightarrow 2^X$ which is fixed point-free on ∂U. From Proposition 2.6 we infer $\gamma_0 \in \Gamma$ and $V_0 \in \mathcal{V}(X)$ such that for all $\gamma \geq \gamma_0$ and $(\psi_\gamma, \varphi_\gamma) \in (F, \gamma)$

$$Fix(\psi_\gamma\varphi_\gamma) \cap V_0(\partial U) = \emptyset.$$

Hence, in particular $(\varphi\psi)|_{\psi^{-1}(\overline{U})}$ is fixed point-free on $\partial_{P_\gamma}\psi^{-1}(U)$, where P_γ is the convex polyhedron associated to $(\psi_\gamma, \varphi_\gamma)$. Thus we can consider the topological index, see e. g. [EF78, Kapitel 7],

$$ind(P_\gamma, (\varphi_\gamma\psi_\gamma)|_{\psi^{-1}(\overline{U})}, \psi^{-1}(U))$$

for $\gamma \geq \gamma_0$. Observe that if $U = X$ this index is 1 since P_γ is AR(compact metric) and therefore $ind(P_\gamma, f_\gamma, P_\gamma) = \Lambda(f) = 1$ for any continuous $f_\gamma : P_\gamma \longrightarrow P_\gamma$. See Corollary 1.1 or [EF78, Satz 7.1.3].

The question arises whether is it possible to construct homotopies like the above f_γ^t to assign a well-defined index to $F : \overline{U} \longrightarrow 2^X$.

This problem is related to the question whether it is reasonable to allow more general polyhedra than the convex ones for the lower tips of our triangles 2.1. Additional fixed point theorems come to mind which relie on the calculation of the index. See e. g. [Bou57].

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Affirmation

Hereby I affirm that I wrote the present thesis without any inadmissible help by a third party and without using any other means than indicated. Thoughts that were taken over directly or indirectly from other sources are indicated as such. This thesis has not been presented to any other examination board in this or a similar form, neither in this or any other country.

The present thesis was written at Dresden University of Technology under the supervision of Prof. Dr. Th. Riedrich.

Versicherung

Hiermit versichere ich, daß ich die vorliegende Arbeit ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe; die aus fremden Quellen direkt oder indirekt übernommenen Gedanken sind als solche kenntlich gemacht. Die Arbeit wurde bisher weder im Inland noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die vorliegende Arbeit wurde an der Technischen Universität Dresden unter der Betreuung von Herrn Prof. Dr. Th. Riedrich angefertigt.

Dresden, 12.2.2001

Thomas Okon